

# Pricing & Risk Management of Synthetic CDOs

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## Abstract

The purpose of this paper is to analyze the risks of synthetic CDO structures and their sensitivity to model parameters. In order to measure these sensitivities, I also introduce the latest techniques in the pricing and risk management of synthetic CDOs. I show how to model the conditional and unconditional default distributions of a typical synthetic deal using a simple mathematical framework. Strictly speaking, the findings of this paper are only directly applicable to synthetic structures, however; many of the modeling and risk-management insights discussed apply to structures involving a waterfall.

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## **Introduction**

The purpose of this paper is to introduce the latest techniques in the pricing and risk management of synthetic CDOs and measure the risks and model sensitivities of a typical synthetic deal. Although these techniques are only directly applicable to synthetic structures, many of the modeling and risk-management insights discussed in the paper apply to structures involving a waterfall.

A synthetic collateralized debt obligation, synthetic CDO, is made up of a portfolio of credit default swaps. The arranger of a synthetic CDO distributes the credit risk of the portfolio by creating and selling tranches to investors. Every tranche has an attachment and detachment point that determines the amount of loss, and correspondingly the number of defaults, the tranche can absorb. For example, the first tranche, known as the equity tranche, might be responsible for portfolio credit losses between 0% and 3%, the next tranche would then be responsible for portfolio losses that exceed 3% up to the size of the tranche, and so on. The least risky tranche of a CDO is known as the senior tranche, or super-senior tranche. Tranches between the equity tranche and the most senior tranche are known as mezzanine tranches.

The challenge in pricing a synthetic CDO lies in the difficult task of formulating a model for the joint default behavior of the underlying reference assets. Understanding and modeling the joint default dynamics of the reference assets are important in order to compute the expected losses for each tranche. The expected losses, in turn, determine the fair spread of the tranche. In fact, once the joint default distribution of the reference assets has been specified, we can price any tranche that references these assets.

In the following section, I will present a simple approach to computing the joint default distribution of a reference portfolio. The approach is based on a recursive procedure and requires no Monte Carlo simulations. I will also compare the results of this recursive approach with the results obtained using a Monte Carlo procedure that simulates the default times of the reference assets and the corresponding losses in the portfolio. The Monte Carlo approach is computationally time-consuming as it requires a large number of simulations in order to produce enough defaults that can impact the most senior tranches of a CDO.

## **Computing the Distribution of Default Losses**

Pricing a synthetic CDO boils down to computing the joint distribution of defaults of the reference portfolio. Computing the default distribution, in turn, depends crucially on the default probabilities of the reference credits and the pairwise correlation between every pair of credits. The correlation among the assets will drive the joint default behavior of the assets. The model we use here is a one-factor model whereby the defaults are driven by one factor which we take to represent a common economic driver of credit events. Default losses are then calculated conditional on the state of this economic factor. This procedure will result in computing the conditional default distribution. The next step is to integrate the conditional default distribution over the common factor to arrive at the

unconditional distribution of default losses. This modeling framework has an appealing and easy interpretation. Conditional on the state of the common economic factor, credits will default when their asset values fall below a pre-specified threshold. This default threshold usually represents the level of debt of a company. If we further assume that the variables driving the returns process follow a normal distribution, then this modeling framework is also known as the Gaussian copula.

We assume that the reference portfolio contains  $N$  credits and each credit is described by its notional amount, probability of default, and recovery rate. For any credit “ $i$ ” we then have a notional  $A(i)$ , a default probability  $p(i)$ , and a recovery rate  $r(i)$ . The return process of each credit is driven by a common factor  $M$ , and a noise factor  $\varepsilon(i)$  that is specific to the  $i$ -th credit according to the following equation:

$$Z(i) = \beta(i).M + \varepsilon(i).\sqrt{(1 - \beta(i)^2)} \quad (1)$$

Where the  $Z(i)$  represents the returns of credit  $i$ . The market factor  $M$ , and the idiosyncratic factor  $\varepsilon(i)$  are independent standard normals with zero means and unit variances. The asset returns,  $Z(i)$ , follow a standard normal distribution as well. Within this specification of the returns dynamics, the correlation between any two credits,  $i$  &  $j$ , is simply given by the product of  $\beta(i)$  and  $\beta(j)$ , where  $\beta(i)$  and  $\beta(j)$  are taken to represent the betas of credits  $i$  and  $j$  respectively. That is, they represent the sensitivities of credits  $i$  and  $j$  to changes in the common factor.

$$\rho(Z(i), Z(j)) = \beta(i).\beta(j) \quad (2)$$

Conditional on the realizations of the common factor, defaults are only driven by the noise factors  $\varepsilon(i)$  and are thus independent. A credit,  $i$ , is assumed to default if its asset return,  $Z(i)$ , falls below a pre-specified level or default threshold given by the  $\Phi^{-1}(p(i))$  where  $\Phi^{-1}$  denotes the inverse of the cumulative standard normal distribution. If we denote the default threshold of credit  $i$  by  $D(i)$ , then a credit  $i$  defaults when:

$$Z(i) < D(i), \text{ where } D(i) = \Phi^{-1}(p(i)) \quad (3)$$

Equivalently by re-arranging equation (1), default occurs when:

$$\varepsilon(i) < [D(i) - \beta(i).M] / \sqrt{(1 - \beta(i)^2)} \quad (4)$$

Finally, since  $\varepsilon(i)$  are standard normals and we assume a flat correlation across all credits, the default probability of credit  $i$  conditional on the realizations of the common factor  $M$  is given by:

$$\text{Prob}(Z(i) < D(i) \mid M) = \Phi([D(i) - M.\beta(i)] / \sqrt{(1 - \beta(i)^2)}) \quad (5)$$

This last equation demonstrates that only a single correlation parameter  $\rho$  and a single common factor  $M$  are needed to calculate the joint distribution of default losses.

There are two ways to move on from here. One approach involves a Monte Carlo simulation of  $M$  and  $\varepsilon(i)$  in equation (1) to generate realizations of the asset returns  $Z(i)$ . Defaults will then be triggered whenever  $Z(i)$  falls below a threshold as described by equation (3). The term structure of default probabilities for each credit can be calibrated to market spreads or implied from the credit ratings. A second approach takes advantage of the fact that defaults are independent, conditional on the common market factor. It then uses a recursive method to construct the conditional default distribution. The details of this recursion method are discussed in Gibson (2004), Hull & White (2003), and Andersen, Sidenius and Basu (2003) and an alternative approach using generating functions is discussed in Mina and Stern (2003). We can calculate the conditional probability that a portfolio of size  $i$  will lose exactly  $k$  credits by time  $t$  with the following recursion:

$$P^i(k, t | M) = P^{i-1}(k, t | M) \cdot (1 - P(Z(i) < D(i) | M)) + P^{i-1}(k-1, t | M) \cdot P(Z(i) < D(i) | M) \quad (6)$$

By starting with a portfolio of size 0 and successively adding credits according to the recursion equation we can construct the conditional default distribution. Finally, by weighting the conditional probabilities by the probability distribution of the common factor we arrive at the unconditional default distribution.

For the results in this article I use the Monte Carlo and the recursion methods to generate the default loss distribution. The Monte Carlo method, though slower, allows more flexibility in modeling the correlation and default parameters. When I run both models using the same correlation and default assumptions, I get the same results.

### **Pricing Synthetic CDO Tranches**

Pricing a CDO tranche is a function of the tranche's notional, spread, and expected default losses. The expected losses on a tranche can be estimated from the default distribution of the reference portfolio. Thus, for each payment date we need to estimate the credit losses sustained by the portfolio and distribute these losses to each tranche based on the relative position of the tranche in the capital structure: the protection leg. Also, each tranche receives a premium that is a function of the remaining notional amount of the tranche on the payment date: the premium leg. Thus, both the premium leg and the protection leg are a function of a common denominator: the portfolio credit losses sustained by the payment date. To compute the fair spread on a tranche we need to equate its premium leg to its protection leg, which reduces the pricing of a synthetic tranche to the more familiar analytics of a single-name default swap. If we denote the expected loss of a tranche at payment date  $t_i$  by  $E(L_i)$ , then:

$$\text{Total Expected Losses on the tranche} = \sum_i D_i \cdot [E(L_i) - E(L_{i-1})], \quad (7)$$

$$\text{Total Expected Premium Payments} = s \cdot \sum_i D_i \cdot E(N_i) \quad (8)$$

Where  $D_i$  is the discount factor at payment date  $i$ ,  $s$  is the tranche spread,  $E(N_i)$  is the tranche expected remaining notional by payment date  $i$ , and the summation is taken over

all payment dates. I should also note that the remaining notional is a function of the expected losses on the tranche, which is driven by the portfolio credit losses. That is:

$$E(N_i) = N_0 - E(L_i), \text{ where } N_0 \text{ is the original notional of the tranche} \quad (9)$$

We can then calculate the fair spread of the tranche as:

$$s_{\text{fair}} = \frac{\sum_i D_i \cdot [E(L_i) - E(L_{i-1})]}{\sum_i D_i \cdot E(N_i)} \quad (10)$$

Once again, the pricing equations show that all one needs to compute the price of any synthetic CDO tranche is the default distribution of the reference credits.

### Pricing & Risk Management of Synthetic CDO Tranches

To illustrate the results of the modeling approach described above, let us work with the following transaction: \$1 billion reference portfolio for a 5-year hypothetical CDO consisting of 100 reference credits. All credits have the same spread of 100 basis points and an average recovery rate of 40%. The flat asset correlation is assumed to be 25%, and the risk-free discount rate is a constant 5%. In addition, all credits have the same notional amount: \$10 million. Table 1 shows the tranches of this hypothetical CDO along with their fair spreads as calculated using the one-factor pricing model.

Table 1  
Pricing of a hypothetical CDO

Tranches	Attachment Point	Detachment Point	Expected Loss %	Fair Spread (bps)	Implied Rating
Equity	0%	3%	65.05	2557	Un-rated
Class D	3%	6%	34.27	923	Caa3
Class C	6%	9%	19.15	465	Caa1
Class B	9%	12%	10.99	255	B2
Class A	12%	22%	3.81	86	Ba2
Senior	22%	100%	0.08	1.8	Aa3
<b>Portfolio</b>	<b>0.00%</b>	<b>100%</b>	<b>4.80%</b>	<b>100</b>	<b>Ba2</b>

As Table 1 shows, the equity and more junior tranches bear the majority of the portfolio credit risk, although they represent a small portion of the capital structure of the CDO. In addition, we can use the expected losses to infer the implied rating of each tranche. The implied ratings show how the credit risk of a Ba2-rated reference portfolio can be distributed as to create buckets of lower and higher quality tranches suitable for various investors. Figure 1 shows the unconditional default probability distribution of the reference credits.

Figure 1  
 Unconditional default probability distributions of a hypothetical CDO

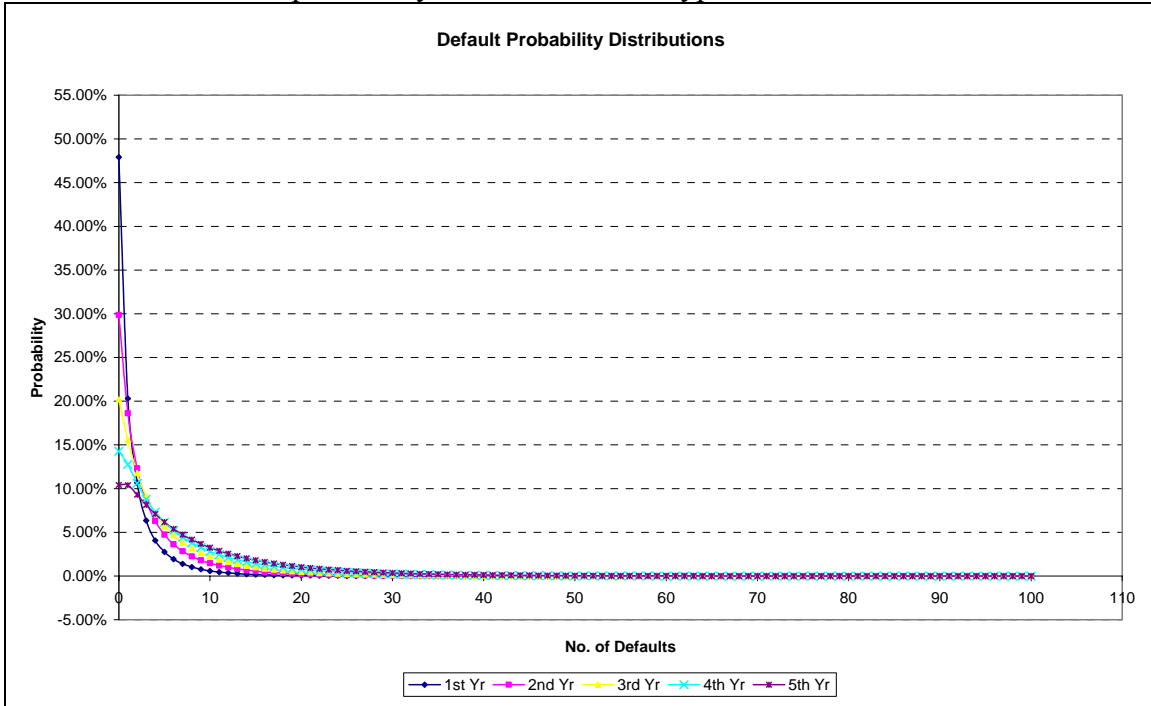


Figure 2 shows the total expected losses on the reference credits conditional on the realizations of the common economic factor. In this graph, the common factor takes values in the interval  $[-5, 5]$  where the negative realizations represent progressively deteriorating market conditions. For example, a value of  $-5$  for the common factor represents a 5-sigma market event. This graph has an intuitive interpretation: lower values of the common economic factor correspond to lower economic growth and higher probabilities of economic recession. Therefore, the expected losses of the portfolio conditional on the economic factor will be higher for lower values of the common factor. The graph can therefore serve as a “scenario” or “what-if” analysis.

Similar graphs can also be produced for each tranche. To illustrate this analysis further, I take the 37<sup>th</sup>, 50<sup>th</sup>, and 63<sup>rd</sup> percentiles of the common factor realizations and calculate the portfolio expected losses at these points. The 37<sup>th</sup> percentile corresponds to a value of  $-1.3$  for the common factor and represents a market downturn, the 50<sup>th</sup> corresponds to a value of  $0$  for the common factor and represents a stable market, and the 63<sup>rd</sup> percentile corresponds to a value of  $1.3$  for the common factor and represents an expanding market. Table 2 shows the results of this scenario analysis. For the sake of completion, figure 3 and table 3 show the conditional expected losses for the equity tranche, class A tranche, and the senior tranche.

Figure 3 clearly shows that the most senior tranche is not totally immune to losses, while the equity tranche bears a substantial risk of default losses even under relatively positive market conditions. In interpreting these results, the reader should bear in mind that we are starting with a Ba2-rated reference portfolio.

Figure 2

Conditional distribution of expected losses for the reference credits over 5 years

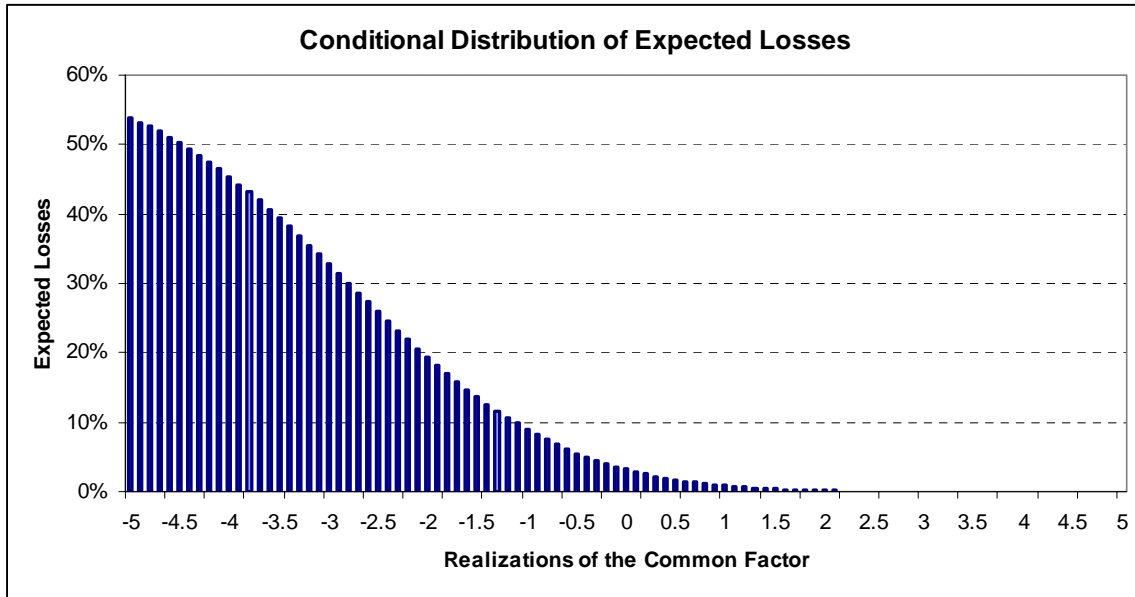


Table 2

Scenario analysis of conditional expected losses over 5 years

Scenario Analysis of Expected Losses	The States of the Economy		
	Downturn	Stability	Growth
Expected Losses of the Portfolio	11.63%	3.20%	0.54%

Figure 3

Conditional expected losses for selected tranches over 5 years

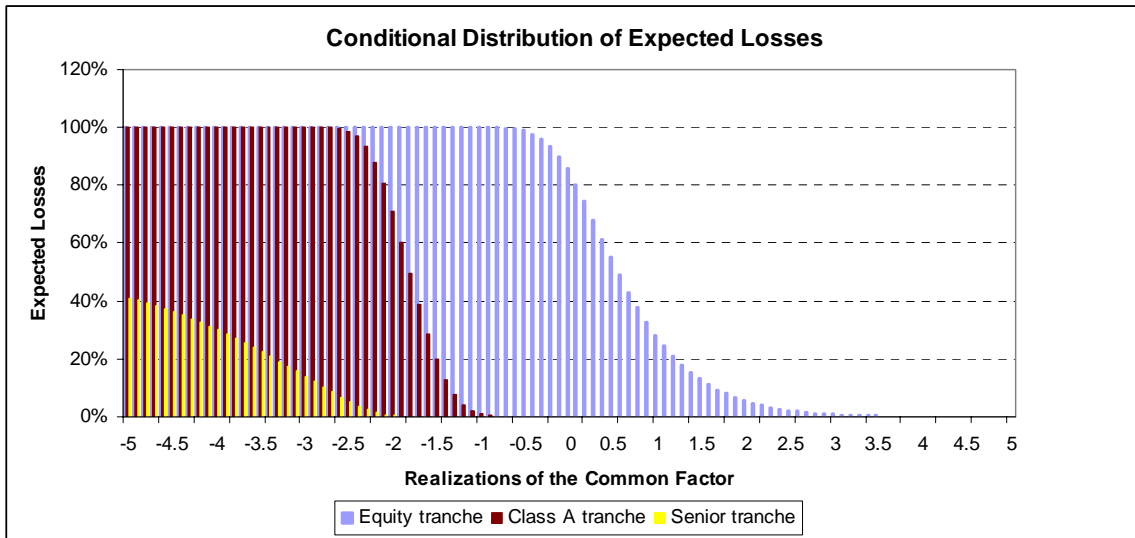


Table 3

Scenario analysis of conditional expected losses for selected tranches over 5 years

Scenario Analysis of Expected Losses	The States of the Economy		
	Downturn	Stability	Growth
Expected Losses of Equity tranche	100%	85.6%	18.0%
Expected Losses of Class A	7.74%	0.00%	0.00%
Expected Losses of Senior tranche	0.00%	0.00%	0.00%

As discussed in Gibson (2004), table 3 illustrates how the mezzanine tranches can be thought of as leveraged bets on business cycle risk. Investors in mezzanine tranches receive spreads ranging from 923 to 86 bps according to table 1; however, they have to absorb the majority of credit risk in difficult and recessionary market conditions.

### Parameter Sensitivities of Synthetic CDO Tranches

In addition to calculating the conditional and unconditional expected losses of the tranches, we can also extend the risk analysis of a synthetic CDO tranche to include:

- Computing the tranche sensitivity to changes in correlation.
- Computing the tranche sensitivity to broad changes in credit spreads.
- Computing the change in subordination necessary to maintain the base value of a tranche as a function of the average credit quality of the reference portfolio.
- Computing the standard deviation of losses.

#### A. Correlation Sensitivity

Figure 4 shows the correlation sensitivity of the equity and mezzanine tranches. The graph shows the fair spread of each tranche at different correlation assumptions as a multiple of the base spread at 25% correlation. The equity tranche is clearly long correlation as its value increases with higher correlations: the spread falls as correlation increases. This is typical for an equity tranche because higher correlation increases the probability of fewer defaults as well as the probability of more defaults. Since equity investors are sensitive to any default it makes sense that they would prefer higher probabilities of fewer defaults - hence higher correlation. In contrast, Class A investors, 12-22% tranche, are short correlation. For Class A investors, higher correlation reduces the value of the tranche and increases its spread.

Tranches in the middle of the capital structure share similar behavior with either the equity tranche or the more senior tranches, but with much less sensitivity to correlation assumptions. For example, Class C, 6-9% tranche, shows very little sensitivity to changes in correlations far beyond the initial assumption of 25% correlation. Figure 5 completes the picture by showing the high sensitivity of the most senior tranche to changes in correlation. More senior tranches of a CDO transaction are only susceptible to extreme market shocks that cause higher market correlations and multiple defaults to occur.



These observations are consistent with the scenario analysis shown in table 3 and figure 3 which illustrate the conditional impact of wide economic downturns on the value of senior tranches. In fact, although not shown table 3, at the 24<sup>th</sup> percentile value of the common default driver, Class A is expected to lose all its notional and the most senior tranche is expected to suffer a 7% loss. Economic shocks of such a magnitude are not unheard of.

The correlation sensitivity analysis illustrates two important features of CDO investing:

- Investors with different correlation assumptions will attach different values to the same tranche. This creates both model risk and an opportunity for correlation and/or model arbitrage. Correlation is a very difficult parameter to measure and estimates of correlation are susceptible to estimation errors, personal judgments, and correlation breakdowns among many others.
- Mezzanine tranches are the least sensitive to changes in the correlation parameter. Therefore, these tranches are also the least sensitive to modeling errors. For example, Class C investors will notice very little change to the value of their tranche even if correlation doubles. Investors who wish to minimize parameter risk will therefore prefer the middle tranches of a CDO transaction.

Figure 4  
Correlation sensitivity of CDO tranches: base spread is calculated at 25% correlation

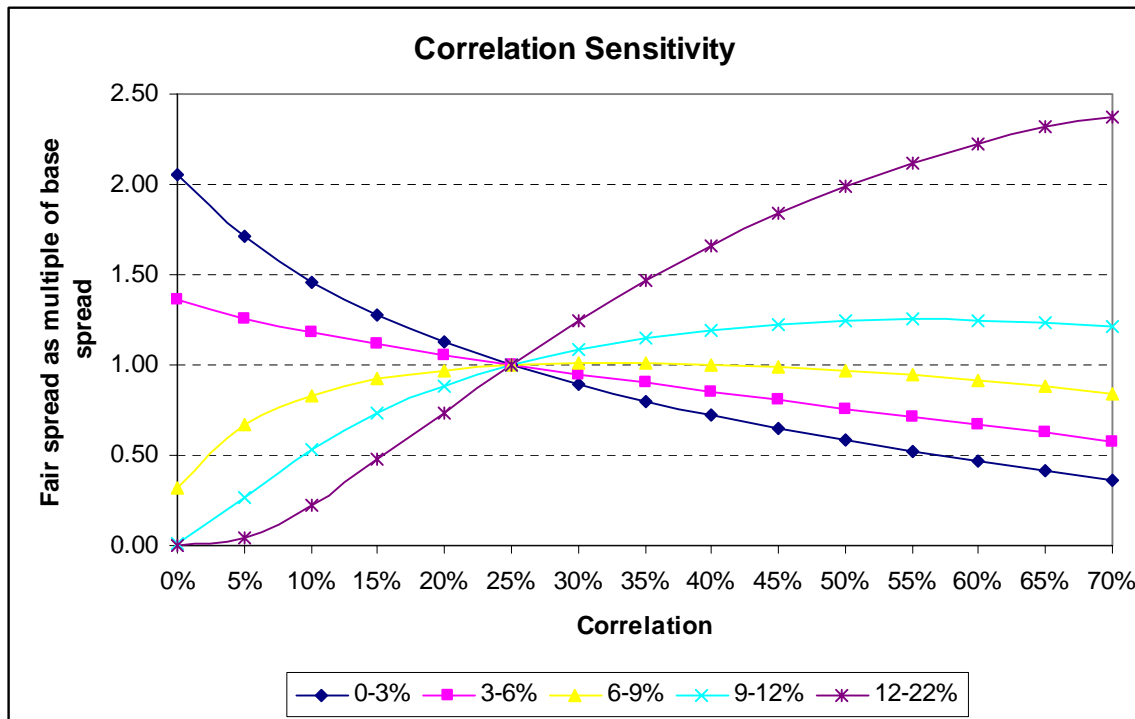
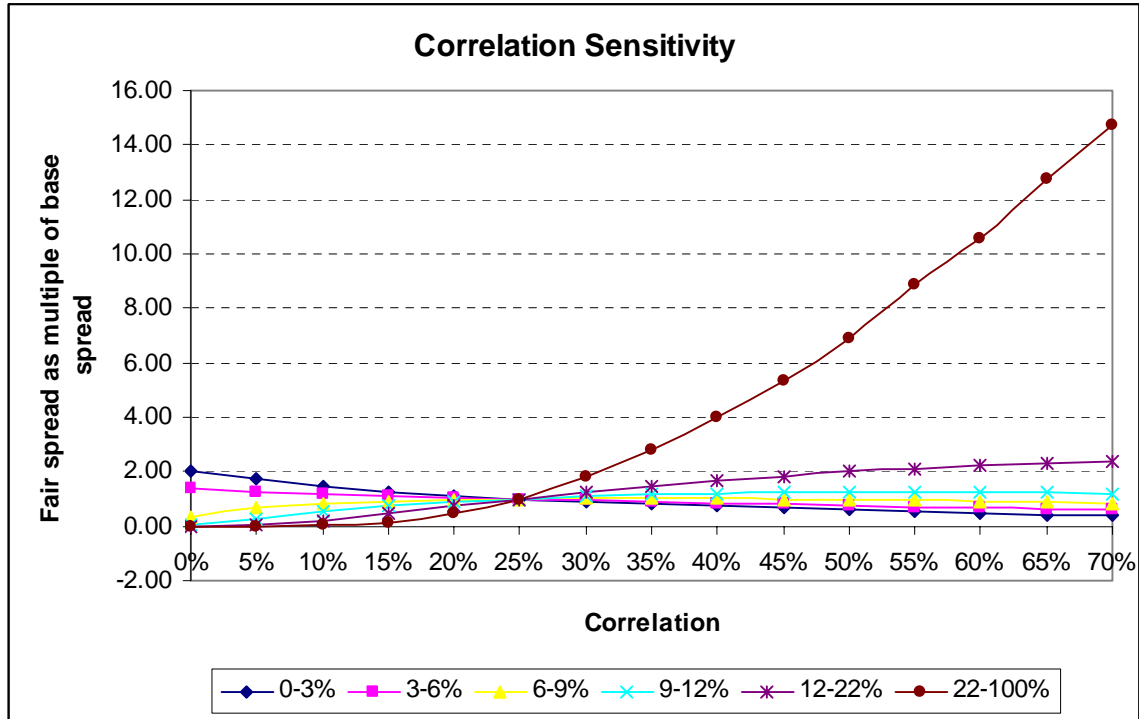


Figure 5  
 Correlation sensitivity of CDO tranches: base spread is calculated at 25% correlation



### B. Sensitivity to Broad Spread Changes

To see the effect of broad economic changes in credit spreads on CDO tranches, I calculate the tranche fair spreads after increasing the credit spreads by 10 bps for all credits. I also calculate the decrease in each tranche’s mark-to-market value and report the results in dollar terms and as a percentage of tranche notional. Table 4 reports these results. The spread on the equity tranche is the least sensitive to a broad increase in credit spreads. However, the change in the mark-to-market as a percentage of tranche notional is greatest for the equity tranche. In contrast, the spread on the most senior tranche has the greatest sensitivity to broad changes in market spreads, though its MtM is least impacted. These results are expected since the increase in credit spreads will increase the likelihood for a higher number of credit defaults to which the senior tranches are more sensitive. That is, in a market environment where credit quality is deteriorating, the probability that we will see multiple credit defaults will increase, which will lower the credit subordination available to the senior tranches, and increase the probability of credit losses reaching the senior tranches. It should be noted here that the effect of a broad increase in default probabilities across credits is indistinguishable from the effect of an increase in the default correlation among credits.

Table 4

Tranche sensitivity to a 10-bps increase in spreads of all credits.

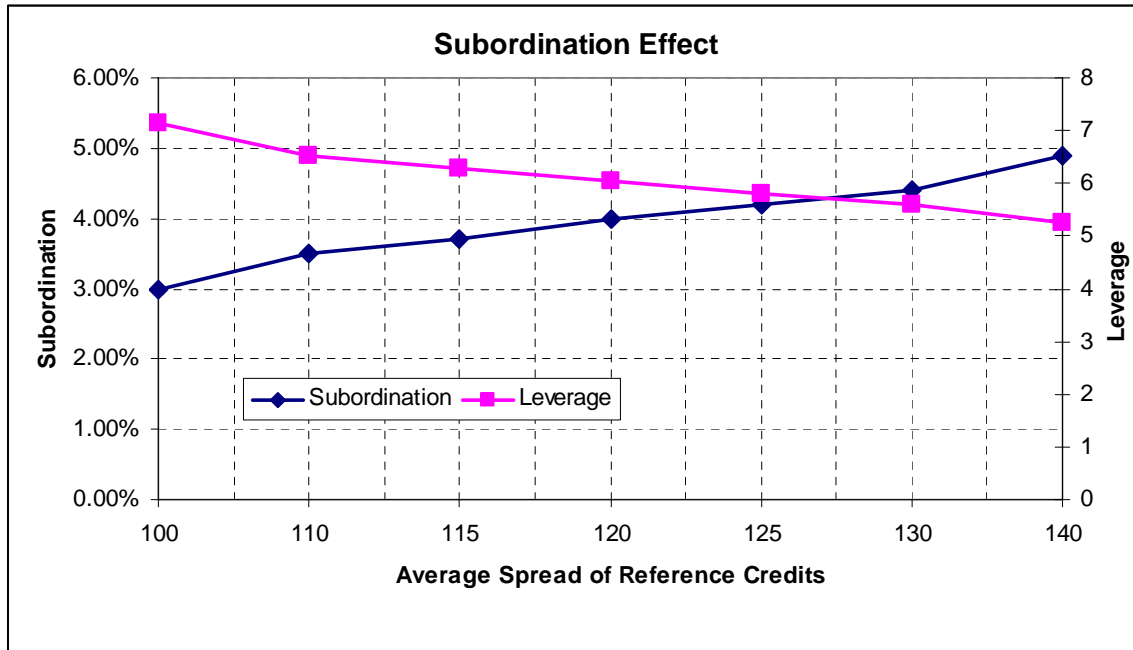
Tranche	Base Spreads	New Spreads	Spread Change	MtM Change \$	MtM Change
0-3%	2557	2781	8.76%	1,635,373	5.45%
3-6%	923	1035	12.13%	1,222,732	4.08%
6-9%	465	535	15.05%	848,075	2.83%
9-12%	255	301	18.04%	581,871	1.94%
12-22%	86	105	22.09%	822,501	0.82%
22-100%	1.8	2.4	33.33%	215,233	0.03%

### C. The Subordination Effect

The expected loss of a tranche is driven not only by the credit spreads of the reference assets but also by the credit enhancement available to the tranche. To illustrate the effect of subordination, or location of a tranche within the capital structure, I run the following analysis: the original reference portfolio is divided into two equal groups. In the first group, the reference credits retain their initial spreads of 100 bps, while the second group's credit spreads are varied to 120, 130, 140, 150, and 160 bps. Starting with the base case of a 100 bps for all credits, I calculate the expected loss on Class D, the 3-6% tranche. Then I change the reference portfolio so that half the credits have a spread of 120 bps, and I back out the subordination necessary to bring about a similar expected loss for Class D. I repeat the same analysis with 130, 140, 150, 160, and 180 bps to obtain the levels of subordination that will maintain the expected loss of Class D at the same base level in each case.

The graph in figure 6 shows how the subordination varies with the average spread on the reference portfolio. As we progressively decrease the quality of the reference portfolio, we need higher levels of subordination to maintain the expected loss of Class D at a level similar to its base level of 34.27%. This analysis illustrates another subtlety in CDO structuring: practically any desirable rating can be attained for a tranche provided the right amount of credit enhancement can be provided to that tranche. The lower the credit quality of the reference portfolio, the more subordination the tranche will require to achieve the same rating. Figure 6 also shows the effect of higher subordination on tranche leverage. Leverage is defined as the expected loss of the tranche divided by the expected loss of the reference portfolio. As figure 6 shows, higher subordination leads to lower leverage. In other words, the lower the credit quality of the reference portfolio, the lower the leverage a tranche will require to achieve a particular rating.

Figure 6  
Subordination effect on expected losses and leverage



#### D. Measuring Standard Deviation and Unexpected Losses

The one-factor Gaussian model allows an easy and straightforward calculation of the standard deviation of tranche losses. Given the expected loss of a tranche at time T, its loss distribution, and its default probabilities, we can calculate the standard deviation of losses at time T with this formula:

$$\text{Standard Deviation} = [\sum_k (x - EL)^2 \cdot P(k, T)]^{0.5} \quad (11)$$

Where x is the tranche loss given the number of defaults in the reference portfolio and the summation is taken over 0 defaults to k = 100 defaults. Using equation 11, I calculate the standard deviation of losses for each tranche. The results are shown in table 5. We can then use the standard deviation to define measures for unexpected losses and required economic capital to hedge against extreme losses. Here I define the unexpected losses of a tranche to be the loss level at one standard deviation above the expected loss of the tranche

Table5  
Measures of tranche risk and unexpected losses

	Portfolio	0-3%	3-6%	6-9%	9-12%	12-22%	22-100%
S. Deviation	5.3%	37%	45%	39%	31.7%	17.34%	1.07%
S. Deviation (\$ mil)	52.9	11.1	13.5	11.7	9.5	17.3	8.3
Unexpected Loss (\$ mil)	96.2	30.0	23.7	17.5	12.8	21.1	8.9
Percent of Notional	9.62%	100%	80%	58%	43%	21%	1.15%

## Monte Carlo Simulation of Losses

The Monte Carlo approach, described in Li (2000), generates the loss distribution by simulating the default times of the reference credits using the Gaussian copula. This approach is flexible in that it allows for stochastic modeling of the recovery and default parameters. The Monte Carlo approach can easily incorporate more than one economic factor, it allows for sampling losses from fat-tailed distributions, and it is capable of capturing a more complex correlation structure. However, the Monte Carlo approach is time-consuming as it requires a large number of simulations in order to reduce estimation error and capture extreme losses that will only affect the most senior tranches. The recursive approach, on the other hand, is simpler and faster, but lacks the flexibility of the Monte Carlo approach.

Using the same modeling assumptions, the Monte Carlo approach should produce similar results to those obtained from the recursion method. That is, if we set the recovery rates, credit spreads, and correlation parameter in the Monte Carlo model to those used for the recursion method, the results of both approaches ought to be similar. Table 6 shows the Monte Carlo results using one million simulations. As evident from table 6, the recursion approach provides a robust and accurate estimation of expected losses and standard deviations.

Table 6:

Expected losses and standard deviations with one million Monte Carlo simulations

	Portfolio	0-3%	3-6%	6-9%	9-12%	12-22%	22-100%
Expected Losses (%)	4.8	66.8	35	19.1	10.6	3.5	0.076
S. Deviations (%)	5.0	37.8	44	37	29.3	15.5	0.009

## Conclusion

In this article, I have demonstrated a simple, yet powerful technique to calculate the default distribution of a credit portfolio. Using this technique, I have shown how the fair spreads and risk parameters of synthetic CDO tranches can be calculated. The important highlights of this article can be summarized as follows:

- The equity and most junior tranches of a synthetic CDO bear the majority of credit risk of the reference portfolio.
- The most senior tranche is not completely immune to credit losses.
- The value of the equity and most junior tranches increases as correlation across the credits rises. On the other hand, the value of the senior tranches decreases with higher correlation.
- The mezzanine tranche is the least sensitivity to the correlation parameter and to correlation model risk.
- The senior tranches are more sensitive to broad changes in credit spreads.
- There is a trade-off between the quality of the reference credits and the subordination required to attain a particular rating for a tranche. The lower the quality of the reference credits, the higher the required subordination.

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