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A Primer on Probability Theory and Stochastic Modelling

Mohamoud Dualeh

May 2004

Markov Chain

Suppose there are two urns. John has one urn and Abdul has the other. The two urns each contain *n* balls. Of the total of 2*n* balls, *n* are red and *n* are black. At each step of a random process, one of the balls in each urn (John's and Abdul's) is chosen at random and John and Abdul then exchange these two balls. So, that each urn continues to contain *n* balls. Let the state of the system be indexed by the number[1], *r*, of red balls in John's urn.

Now, the transition probabilities for a Markov Chain model of this process is given by:

If there are X_i red balls in John's urn at step i,

$$P_{r,s} = P\left(X_{i+1} = s \middle| X_i = r\right) \qquad for\ r,\ s = 0,\ 1,\ \ldots,\ n.$$

$$Then\ P_{0,1} = 1; \quad also\ P_{n,n+1} = 1$$

When $X_i = r$, John's urn contains *r* red and (*n*-1) black, and Abdul's (*n*-1) red and *r* black.

$$P_{r,r+1} = P\left(choose\ red\ in\ Rod's\ urn \middle| X_i = r\right).\ P\left(choose\ \ black\ in\ Abukar's\ urn\ \middle| X_{i=r}\right)$$

$$= \left(\frac{r}{n}\right)^2$$

$$P_{r,r} = 2\left(\frac{r}{n}\right)\left(1-\frac{r}{n}\right) \quad by\ similar\ argument;\ and\ also$$

$$P_{r,r+1} = \left(1-\frac{r}{n}\right)^2 \quad otherwise\ P_{r,s} = 0$$

Now that we worked out an expression for transition probabilities, we can write a system of equations that must be satisfied by the stationary distribution of this model.

$$\overline{\Pi} = [\Pi_0, \Pi_1, \ldots, \Pi_n]$$

The probability π_i *is* $P(i\ red\ balls\ in\ Rod's\ urn)$.

Therefore,

$$\pi_0 = \left(\frac{1}{n}\right)^2 \pi_1; \quad and\ \pi_n = \left(\frac{1}{n}\right)^2 \pi_{n-1}. \qquad \text{(A)}$$

[1] You can choose your own index if you prefer.

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for r = 1, 2, n-1,

$$\pi_r = P_{r-1,r}\pi_{r-1} + P_{r,r}\pi_r + P_{r+1,r}\pi_{r+1},$$

$$i.e.\ \ \pi_r = \left(1-\frac{r-1}{n}\right)^2 \pi_{r-1} + 2\left(\frac{r}{n}\right)\left(1-\frac{r}{n}\right)\pi_r + \left(\frac{r+1}{n}\right)^2 \pi_{r+1}. \qquad \text{(B)}$$

Equations (A) and (B) define the process, with $\sum_{r=0}^{n}\pi_r = 1$.

Example:

Suppose n = 3, we can solve above equations in order to find Π_1, Π_2 *and* Π_3.

From equation (A) we have

$$\pi_0 = \frac{1}{9}\pi_1$$

$$\pi_3 = \frac{1}{9}\pi_2$$

and $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

Also, from equation (B) we have

$$\pi_1 = \left(1-\frac{1-1}{3}\right)^2 \pi_0 + 2\left(\frac{1}{3}\right)\left(1-\frac{1}{3}\right)\pi_1 + \left(\frac{1+1}{3}\right)^2 \pi_2;\ \textit{substituting gives}$$

$$\frac{1}{9}\pi_1 + \pi_1 + \pi_2 + \frac{1}{9}\pi_2 = 1 = \frac{10}{9}(\pi_1 + \pi_2), \quad so\ \pi_1 + \pi_2 = \frac{9}{10}.$$

Now use

$$\pi_0 = \frac{1}{9}\pi_1$$

$$\pi_2 = \frac{9}{10} - \pi_1$$

$$\pi_3 = \frac{1}{10} - \frac{1}{9}\pi_1$$

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Therefore

$$\pi_1 = \pi_0 + \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2$$

$$= \frac{1}{9}\pi_1 + \frac{4}{9}\pi_1 + \frac{4}{9}\left(\frac{9}{10} - \pi_1\right) = \frac{1}{9}\pi_1 + \frac{2}{5}.$$

$$\therefore \frac{8}{9}\pi_1 = \frac{2}{5},$$

or

$$\pi_1 = \frac{9}{20}$$
$$\pi_0 = \frac{1}{20}$$
$$\pi_2 = \frac{9}{20}$$
$$\pi_3 = \frac{1}{20}.$$

So it is complete. Whew!!

I have omitted couple of tedious calculations! As usual, all the typos and mistakes are mine!

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