Credit Derivative Models: Which Archimedean Copula is the right one?

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CONTENTS

Which Archimedean Copula the right one?

Mario R. Melchiori[2]

Abstract

This paper presents the concept of the copula from a practical standpoint. Given the widened use of the multinormal distribution, we argue its inadequacy, while advocate using the copula as an alternative and better approach, for instance with regard to credit market valuation and pricing. We examine what the copulas are used for within areas of risk management. Then we expose a guide to choose both the margins and the Archimedean copula that provide a better fit to real-world data. In addition, we provide an algorithm to simulate a random bivariate distribution from an Archimedean copula. In order to cover the gap between the theory and its practical implementation, we provide the VBA codes required. We also illustrate the use of copulas in the pricing of a first-to-default credit derivative contract. They are used in a numerical example that illustrates the use of the copula in the pricing of a first-to-default contract. Two spreadsheets accompany the paper, and present a step-by-step description of the practical application of the copula.

Keywords: Copula, Kendall Tau, Dependence, and Credit Derivatives

Introduction

Since Li (2000) first introduced copulas into default modeling, there has been an increasing interest in this approach. Prior to that time, the copula concept was used frequently in survival analysis and actuarial sciences.

Following to Li (2000), a copula is a function that links univariate marginals to their full multivariate distribution. For m uniform random variables $U_1, U_2, ..., U_m$ the joint distribution function C is defined as:

$$C(u_1, u_2, ..., u_m, \rho) = \Pr[U_1 \le u_1, U_2 \le u_2, ..., U_m \le u_m]$$

where ρ [3] is a dependence parameter, can also to be called a *copula function*.

A copula can be used to link marginal distributions with a joint distribution. For determinate univariate marginal distribution functions $F_1(x_1), F_2(x_2), ..., F_m(x_m)$, the function

$$C(F_1(x_1), F_2(x_2), ..., F_m(x_m)) = F(x_1, x_2, ..., x_m)$$

which is defined using a copula function C, results in a multivariate distribution function with univariate marginal distributions specified by $F_1(x_1), F_2(x_2), ..., F_m(x_m)$. Sklar (1959) established the converse. He showed that any joint distribution function F can be seen as a copula function. He proved that if $F(x_1, x_2, ..., x_m)$ is a joint multivariate distribution function with univariate marginal distribution functions , then there exists a copula function $C(u_1, u_2, ..., u_m)$ such that

2 I am grateful to Arcady Novosyolov, Carina Strada, Glyn Holton, Kazuo Oshima, Luciano Alloatti and Moorad Choudhry for their generous contributions. All remaining errors are, of course, my own. I want to thank to Mohamoud B Dualeh for encouraging me to write this paper.

3 As Embrechts et. al. (2001) show, the correlation is only a limited description of the dependence between random variables, except for the multivariate normal distribution where the correlation fully describes the dependence structure.

$F(x_1, x_2, \ldots, x_m) = C(F_1(x_1), F_2(x_2), \ldots, F_m(x_m))$.

If each F_i is continuous then C is unique. Thus, copula functions provide an unifying and flexible way to study joint distributions. Another important derivation is that the copula allows us to model the dependence structure independently from the marginal distributions.

In this paper, we will focus the bivariate copula function $C(u,v)$ for uniform variables U and V, defined over the area $\{(u,v) \mid 0 < u \le 1, 0 < v \le 1\}$

A normal world

Copulas are commonly adopted both in market risk models and credit risk models, either explicitly or implicitly, when the models do use of the multinormal distribution. For instance the commercial credit risk models developed by KMV and CreditMetrics use this.

From the copula's point of view the multinormal distribution has normal marginal distribution and Gaussian copula dependence.

Hereafter, we will use the term *Normal* for the univariate marginal distributions and the term *Gaussian* referring to the copula dependence.

The advantage of using normal dependence structure doesn't arise, as should be suppose, from historical behavior of the financial nor credit market, but in its simplicity, analytical manageability and the easy estimation the its only parameter, the correlation matrix. Empirical evidence suggests that the use of multinormal distribution is inadequate[4]. The non-normality of univariate and multivariate equity returns is historically unmistakable. In other words, there is clear evidence that equity returns have unconditional fat tails, to wit, the extreme events are more probable than anticipated by normal distribution, not only in marginals but also in higher dimensions. This is important both for market risk models as credit risk one, where equity returns are used as a proxy for asset returns that follow a multivariate normal distribution, and, therefore, default times have a multivariate normal dependence structure as well.

As Embrechts *et al* (2001) show, there are many pitfalls to the normality assumption. For us, the main snare is the small probability of extreme joint events. In credit risk case, defaults are rare events, so that the tail dependence has a great impact on the default structure. Tail dependence can be measure. The tail dependence for two random variables X and Y with marginal distributions F_X and F_Y measures the probability that Y will have a realization in the tail of its distribution, conditioned that X has had a realization in its own tail. Tail dependence relates the amount of dependence in the upper right quadrant tail or lower left one of a bivariate distribution, so we could have upper tail dependence, lower tail dependency or both. Upper tail dependence exists when there is a probability that positive outliers happen jointly. Upper tail dependence is defined as:

$$\lambda_{upper} = \lim_{u \to 1} \Pr\left(Y \ge F_Y^{-1}(u) \mid X \ge F_X^{-1}(u)\right) \quad (1.1)$$

where F^{-1} denotes the inverse cumulative distribution function and u is an uniform variable defined over $(0,1)$.

4 See R. Mashal, M. Naldi, and A. Zeevi. **The Dependence Structure of Asset Returns**, forthcoming, RISK.

Since $\Pr\left(Y \geq F_Y^{-1}(u) \mid X \geq F_X^{-1}(u)\right)$ can be written as:

$$\frac{1-\Pr\left(X \leq F_X^{-1}(u)\right)-\Pr\left(Y \leq F_Y^{-1}(u)\right)+\Pr\left(X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(u)\right)}{1-\Pr\left(X \leq F_X^{-1}(u)\right)} \qquad (1.2)$$

given that:

$$\Pr\left(X \leq F_X^{-1}(u)\right)=\Pr\left(Y \leq F_Y^{-1}(u)\right)=u \qquad (1.3)$$

and

$$\Pr\left(X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(u)\right)=C(u,u) \qquad (1.4)$$

an alternative and equivalent definition (for continuous random variables) of (1.1) , is the following:

$$\lambda_{upper} = \lim_{u \to 1} \frac{1-2u+C(u,u)}{1-u} \qquad (1.5)$$

Lower tail dependence is symmetrically defined:

$$\lambda_{Lower} = \lim_{u \to 0} \Pr\left(Y \leq F_Y^{-1}(u) \mid X \leq F_X^{-1}(u)\right) \qquad (1.6)$$

Since $\Pr\left(Y \leq F_Y^{-1}(u) \mid X \leq F_X^{-1}(u)\right)$ can be written as:

$$\frac{\Pr\left(X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(u)\right)}{\Pr\left(X \leq F_X^{-1}(u)\right)} \qquad (1.7)$$

given (1.3) and (1.4) an alternative and equivalent definition (for continuous random variables) of (1.6) , is the following:

$$\lambda_{Lower} = \lim_{u \to 0} \frac{C(u,u)}{u} \qquad (1.8)$$

The Gaussian copula with correlation $\rho < 1$ does not have lower tail dependence nor an upper one $\left(\lambda_{Lower}, \lambda_{upper}\right)$.

It is important to remark that the tail area dependency measure $\left(\lambda_{Lower}, \lambda_{upper}\right)$ depends on the copula and not on the marginal distributions.

Non-Gaussian copulas such as *t* and Archimedean used as underlying dependence structure with anyone else marginal distribution, have upper tail dependence, lower tail dependency or both, so that, they could describe better the reality of the behavior of the financial and credit markets. See the appendix D for a non-parametric estimation of the tail dependence[5].

Credit Derivatives Risk Management Application

Up to now, we have seen what the copula is and why the multinormal distribution is not an adequate assumption. Now, we show what the copula is used for within risk management.

As noted earlier, copulas were used frequently in survival analysis and actuarial sciences. In addition, it is employed in loss aggregation, stress testing, default modelling and operational risk. Hereafter, we concentrate the use of the copula in the default modelling scope, more concretely, in the Credit Derivatives area.

Default risk has been extensively modelled at an individual level, but little is known about default risk at a portfolio level where the default dependence is a meaningful aspect for considering. Further, in recent years we have seen new financial instruments, such as collateralized debt obligations (CDOs), and nth-to-default baskets which have contingent payoffs on the joint default behaviour of the underling securities. In the case of an nth-to-default basket, the joint dependence is of vital importance in its pricing, because the amount of names are not large enough to ensure a correct diversification.

Later in this article we provide an example to illustrate the use of copula in the valuation of a first-to-default contract.

The appropriate choice of the marginal distribution is needed but not enough to accurately measure and price the risk exposure at a portfolio level, in addition is critical to understand and to model the default dependence to choose the fitted joint distribution among the underling securities.

Archimedean copulas

We will focus our attention to one special class of copula termed the Archimedean copula.

An Archimedean copula can be written in the following way:

$$C(u_1,\ldots,u_n)=\varphi^{-1}\left[\varphi(u_1)+\ldots+\varphi(u_n)\right] \tag{1.9}$$

for all $0 \le u_1,\ldots,u_n \le 1$ and where φ is a function termed generator, satisfying:

- $\varphi(1)=0$;
- for all $t \in (0,1), \varphi'(t) < 0$, this is to say φ is decreasing;
- for all $t \in (0,1), \varphi''(t) \ge 0$, this is to say φ is convex.

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5 For a formal calculation of (1.1) see EMBRECHTS, P., A. J. MCNEIL and D. STRAUMANN (1999): **Correlation and Dependence in Risk Management: Properties and Pitfalls**.

Examples of bivariate Archimedean copulas include the following:

- Product or Independent copula:

$$\varphi(t) = -\ln t; C(u,v) = u.v \tag{1.10}$$

- Clayton copula[6]

$$\varphi(\mathrm{t}) = t^{-\theta} - 1, \theta > 0; C\left(u^{-\theta} + v^{-\theta} - 1\right)^{\frac{-1}{\theta}} \tag{1.11}$$

- Gumbel copula[7]

$$\varphi(t) = (-\ln t)^{\theta}, \theta \geq 1; C(u,v) = e^{\left\{-\left[(-\ln u)^{\theta} + (-\ln v)^{\theta}\right]^{\frac{1}{\theta}}\right\}} \tag{1.12}$$

- Frank copula[8]

$$\varphi(t) = -\ln \frac{e^{-\theta t} - 1}{e^{-\theta} - 1}, \theta \in \mathbb{R}; C(u,v) = -\frac{1}{\theta} \ln\left(1 + \frac{\left(e^{-\theta u} - 1\right)\left(e^{-\theta v} - 1\right)}{\left(e^{-\theta} - 1\right)}\right) \tag{1.13}$$

The method described ahead enables one to select the Archimedean copula, which provides a better fit to real-world data. An Archimedean copula has the analytical representation given by equation (1.9). So, in order to select the copula, it is sufficient to identify the generator $\varphi(t)$.

Selecting the right marginal distribution

Suppose you have two historical time series compound by 1000 observed data over a period of time, such as the following:[9]

	Series 1	Series 2
1	0.856617	-0.609474
2	1.221406	0.974876
3	0.359444	1.088642
4	0.777068	0.651016
5	0.734274	0.962609
.	.	.
.	.	.
.	.	.
996	-0.662160	-1.240644
997	-0.567470	-1.196790
998	0.849134	1.456710
999	-0.814523	-0.757466
1,000	0.580571	0.168181

6 Clayton (1978), Cook-Johnson (1981), Oakes (1982).
7 Gumbel (1960), Hougaard (1986).
8 Frank (1979).
9 In the context of this paper these series can be considered as equity returns that are used as a proxy for asset returns. First, we investigate the marginal distribution of each series and then we inquire which is the dependence among them.

First, it is necessary to determine in what manner the series are marginally distributed. For doing this, we using the chart, other data can to request more sophisticated approach[10]. Commercial simulation software such as Crystal Ball[11] or @Risk[12] supply tools for fitting historical data to determinate probability distribution.

	Series 1	Series 2
Average	0.035	0.073
Std Dev	1.021	1.008
Std Err	0.032	0.032
Max	3.234	2.662
Min	-3.247	-2.788
Quantile 95%	-1.675	-1.658

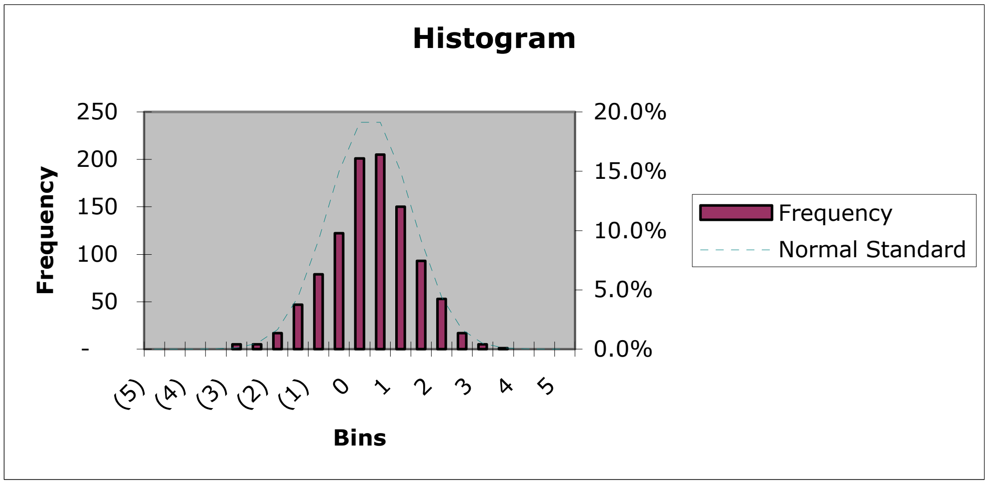
Series 1

Bins	Frequency
-3.00	5
-2.50	5
-2.00	17
-1.50	47
-1.00	79
-0.50	122
0.00	201
0.50	205
1.00	150
1.50	93
2.00	53
2.50	17
3.00	5
3.50	1

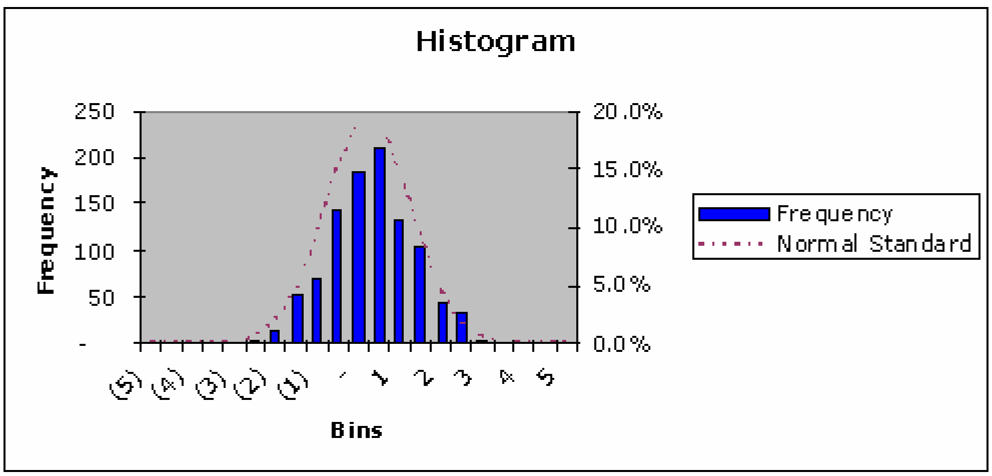
Series 2

Bins	Frequency
-3.50	-
-3.00	-
-2.50	3
-2.00	14
-1.50	53
-1.00	69
-0.50	144
0.00	186
0.50	210
1.00	134
1.50	106
2.00	45
2.50	32
3.00	4
3.50	-

10 A next paper will introduce some of the approach such as Maximum likelihood Estimation
11 **Decisioneering, Inc.** - offers the Crystal Ball line of spreadsheet modeling software for time-series forecasting, risk analysis, and optimization using Monte Carlo simulation.
12 **Palisade Corporation** - develops applications for risk and decision analysis using Monte Carlo simulation and optimization, including @RISK. All are add-ins to Excel



Series 1: Histogram of the historical realization.



Series 2: Histogram of the historical realization.

Data and plots show that the Normal Standard Probability Distribution is a fitted election in this case.

Knowing the marginal distribution, we are able to separate marginal behaviour and dependence structure. The dependence structure is fully described by the joint distribution of uniform variates obtained from the marginal distributions, Normal Standard Distributions in our case. This point is of fundamental importance and often cause considerably troubles. Remember, dependence structure doesn't derive from the marginal distributions, Normal Standard in this example, but from the uniform variates obtained from the marginal distributions. We just need to know marginal distributions so that to recognize the cumulative distribution functions (CDF) that allows us to compute the uniform variate.

For example:

	Series 1	Series 2
1	**0.856617**	**-0.609474**

$$\Phi(0.856617) = 0.804172 \quad (1.14)$$
$$\Phi(-0.609474) = 0.271105 \quad (1.15)$$

where Φ denotes the normal cumulative distribution function. In Excel language:

$$= \text{NormSDist}(0.856617) = 0.804172 \quad (1.16)$$
$$= \text{NormSDist}(-0.609474) = 0.271105 \quad (1.17)$$

the dependence structure refers to the relationship between **0.0804172** and **0.271105**.

Now, what if the marginal distribution is Lognormal? We must perforce then use the correct CDF (LogNormDist function in Excel).

Dependence. Kendall Tau τ

For investigating more deeply the dependence we need a measure for gauging it. It is known as Kendall τ (Tau). It is a rank correlation measure, it is invariant under strictly increasing transformations of the underlying random variables. Linear correlation (or Pearson's correlation (ρ)) is most frequently used in practice as a measure of dependence, but it lacks this property.

If we call c and d respectively the numbers of pairs of variables, which are concordant and discordant, then Kendall's Tau writes :

$$\tau = \frac{c-d}{c+d} = p_c - p_d \quad (1.18)$$[13]

where p_c and p_d are respectively the probabilities of concordance and discordance.

Let $\mathbf{V_t} = [X_t \quad Y_t]$ be a vector of two random variables at time t. In our case $\mathbf{V_{996}} = [-0.662160_{996} \quad -1.240644_{996}]$, for example. Then, two distinct observations $\mathbf{V_t}$ and $\mathbf{V_s}$ are concordant if $(X_t - X_s)\cdot(Y_t - Y_s) > 0$. Conversely, if we have $(X_t - X_s)\cdot(Y_t - Y_s) < 0$, $\mathbf{V_t}$ and $\mathbf{V_s}$ are discordant (i.e. : negatively dependent).

[13] The formula $\tau_n = \binom{n}{2}^{-1} \sum_{i<j} sign\left[(X_{1i} - X_{1j})(X_{2i} - X_{2j})\right]$ (1.32) can be used for estimating τ

Calculating the lineal correlation from both marginal and uniform distribution can see the property of invariability under strictly increasing transformations of the underlying random variables:

Lineal Correlation	
Uniform variables U(0,1)	
1.000	0.655
0.655	1.000
Marginal Distributions	
1.000	0.629
0.629	1.000

They are different. Correlation's Pearson is variant under strictly increasing transformations of the underlying random variables. The fundamental reason why correlation fails as an invariant measure of dependency is due to the fact that the Pearson Correlation coefficient depends not only on the copula but also on the marginal distributions. Thus the measure is affected by changes of scale in the marginal variables.

Now we compute the Kendall Tau dependence:

τ	
Uniform variables U(0,1)	
1.000	0.458
0.458	1.000
Marginal Distributions	
1,000	0,458
0,458	1,000

They are alike. Kendall Tau is invariant under strictly increasing transformations of the underlying random variables.

Archimedean Bivariate Copula

The following algorithm generates random variates $(u,v)^T$ whose joint distribution is an Archimedean copula C with generator φ:

Algorithm:

1. Simulate two independent $U(0,1)$ random variates s and q.
2. Set $t = K_{Copula}^{-1}(q)$, where K_{Copula} is the distribution function $C(u,v)$.
3. Set $u = \varphi^{-1}(s.\varphi(t))$ and $v = \varphi^{-1}((1-s)\varphi(t))$.

For each Archimedean copula we need, to wit:

A. Kendall τ (1.32)
B. Theta θ
C. Generator $\varphi(t)$
D. Generator's first derivate $\varphi'(t)$

E. Generator's Inverse $\varphi^{-1}(t)$

F. The distribution function of $C(u,v) = K_{Copula} = t - \frac{\varphi(t)}{\varphi'(t)}$

G. Distribution function inverse K_{Copula}^{-1} (When it has not a closed form as in case of Gumbel, Frank and Clayton Archimedean copula, it can be obtained through the equation $\left(t - \frac{\varphi(t)}{\varphi'(t)}\right) - q$ by numerical root finding). For doing this, we need the first derivate regard to t of $t - \frac{\varphi(t)}{\varphi'(t)}$.

For the **Gumbel** copula, we have:

Table 1

B.(*)	C.	D.	E.	F.	G.(**)
$\theta = \frac{1}{1-\tau}$ (1.19)	$(-\ln t)^{\theta}$ (1.20)	$-\theta(\ln t)^{\theta-1}\frac{1}{t}$ (1.21)	$e^{\left(-t^{\frac{1}{\theta}}\right)}$ (1.22)	$t - \frac{(t\ln t)}{\theta}$ (1.23)	$-\frac{\ln(t)}{\theta} - \frac{1}{\theta} + 1$ (1.24)

(*) $\theta \geq 1$. Only positive dependence.

(**) There is not a closed form for the inverse distribution function K_{Gumbel}^{-1}, so G. will be used for obtaining it by numerical root finding.

So that:

$$u = e^{-\left(s\cdot(-\ln(t))^{\theta}\right)^{\frac{1}{\theta}}} \quad (1.25)$$

$$v = e^{-\left((1-s)\cdot(-\ln(t))^{\theta}\right)^{\frac{1}{\theta}}} \quad (1.26)$$

VBA code that generates random variates from the 2-dimensional Gumbel copula

Function GumbelCopula(ByVal Theta As Double, Optional Random1, Optional Random2) As Variant

‘ Generates random variates from the 2-dimensional Gumbel copula

Dim t As Double, s As Double, q As Double, u() As Double
Application.Volatile
ReDim u(1 To 2)

‘ Simulate two independent $U(0,1)$ random variates s and q.

```
If IsMissing(Random1) Then
   s = Rnd
   Else
   s = Random1
   End If
   If IsMissing(Random2) Then
```

q = Rnd
Else
q = Random2
End If

‘ Set $t = K_{Copula}^{-1}(q)$ **, where** K_{Copula} **is the distribution function of** $C(u,v)$ **.**

‘ Because Gumbel has not a closed form KCg_Inv is obtained through by numerical root finding
t = KCg_Inv(Theta, q)

‘ Set $u = \varphi^{-1}(s.\varphi(t))$ **and** $v = \varphi^{-1}((1-s)\varphi(t))$**.**
*u(1) = Exp(-(s * (-Log(t)) ^ Theta) ^ (1 / Theta))*
*u(2) = Exp(-((1 - s) * (-Log(t)) ^ Theta) ^ (1 / Theta))*

‘ The vector u(2) is a pair of pseudo random numbers that are uniformly distributed on [0,1] x [0,1] and that has a Gumbel copula as a joint distribution function.
GumbelCopula = u

End Function

VBA Code that computes Inverse Cumulative Distribution Function using numerical root finding

---

Function KCg_Inv(ByVal Theta As Double, ByVal q As Double, Optional tolerance As Single = 0.0000000001) As Double

‘Because Gumbel has not a closed form $t = K_{Copula}^{-1}(q)$ **is obtained through by numerical root finding**

Dim t As Double, tzero As Double, KCg As Double, delta As Double, diff As Double

t = tolerancia
tzero = 0
Do While True

‘ The distribution function of $C(u,v) = K_{Copula} = t - \frac{\varphi(t)}{\varphi'(t)}$ **.Gumbel equal to** $t - \frac{(t\ln t)}{\theta}$ (1.23)

*KCg = t - (t * Log(t) / Theta) – q*

‘ Derivate of the distribution function of $C(u,v) = K_{Copula} = t - \frac{\varphi(t)}{\varphi'(t)}$ **.Gumbel equal to**

$-\frac{\ln(t)}{\theta} - \frac{1}{\theta} + 1$ (1.24)

delta = -(Log(t) / Theta) - (1 / Theta) + 1

diff = KCg - tzero

```
    If Abs(diff) < tolerance Then Exit Do
    t = t + (-diff / delta)
  Loop
  KCg_Inv = t
  Exit Function
End Function
```

The vector u(2) is a pair of pseudo random numbers that are uniformly distributed on [0,1] x [0,1] and it has a Gumbel copula as a joint distribution function.

Then take the marginal distribution functions, in this case, normal standard, we put

$$u = \Phi(r_1) \qquad (1.27)$$
$$\text{v=NormSDist}(\text{r}_1)$$

$$v = \Phi(r_2) \qquad (1.28)$$
$$\text{v=NormSDist}(\text{r}_2)$$

then we have:

$$r_1 = \Phi^{-1}(u) \qquad (1.29)$$
$$r_1 = \text{NormSInv}(\text{u})$$

$$r_2 = \Phi^{-1}(v) \qquad (1.30)$$
$$r_2 = \text{NormSInv}(\text{v})$$

are pseudo random numbers with distribution function Φ (Normal Standard) and joint distribution function Gumbel.

Which Archimedean copula is the right one?

The distribution function of an Archimedean copula, as it already had been exposed in F, is represented for the following formula:

$$C(u,v) = K_{Copula} = t - \frac{\varphi(t)}{\varphi'(t)} \qquad (1.31)$$

To identify φ, we:

1. Estimate Kendall's correlation coefficient using the usual nonparametric estimate:

$$\tau_n = \binom{n}{2}^{-1} \sum_{i<j} sign\left[(X_{1i} - X_{1j})(X_{2i} - X_{2j})\right] \qquad (1.32)$$

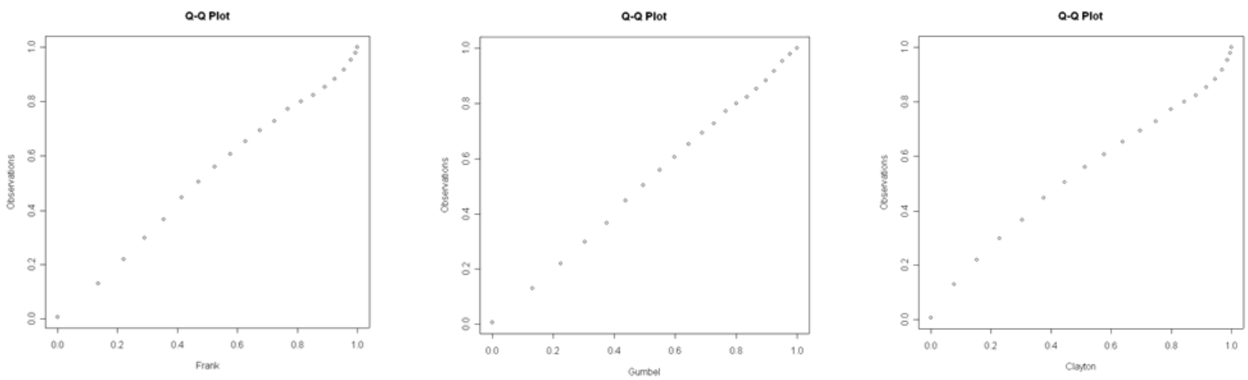
2. Construct a nonparametric estimate of K_{Copula}, the following way:
   i. First, define the pseudo-observations $T_i = \{$number of $(X_{1j} < X_{1i})$ such that $X_{1j} < X_{1i}$ and $X_{2j} < X_{2i}\} \;/\; (n-1)$ for $i = 1,2,...,n$.
   ii. Second, construct the estimate of K_{Copula} as $K_{Copula_n}(t) =$ proportion of T_i's $\leq t$.

3. Now construct a parametric estimate of K_{Copula} using the relationship (1.31)˙

For example, choose a generator φ, for this refers to Table 1 and Appendix A, and use the estimate τ_n to calculate an estimate de θ, say θ_n. Use θ_n to estimate $\varphi(x)$, say $\varphi(x_n)$.Finally, use $\varphi(x_n)$ to estimate $K_{Copula}(t)$, say $K_{Copula_n}(t)$.

In order to select the Archimedean copula which fits better the data, Frees and Valdez (1998) propose to use a Q-Q plot between 2.ii) and 3) or by minimizing a distance such as $\int \left[K_{Copula_n}(t) - K_n(t)\right]^2 dK_n(t)$.

Both approaches are presented below:



The graphical approach shows that the Gumbel copula is the better fit.

The nonparametric approach arrives at the same result.[14]

[14] An attached Excel ™ sheet develops the nonparametric method thoroughly. Over there we only present its result.

	Gumbel	Clayton	Frank
τ	0.45835	0.45835	0.45835
θ	1.84623	1.69245	5.02757

tau Frank

$$\int \left[K_{Copula_n}(t) - K_n(t) \right]^2 dK_n(t)$$

t	Density	Cumulative	Sample	Gumbel	Clayton	Frank	Gumbel	Clayton	Frank
0.00001	7	7	0.00700	0.00007	0.00002	0.00011	0.000	0.000	0.000
0.05001	123	130	0.13000	0.13115	0.07937	0.13513	0.000	0.003	0.000
0.10001	90	220	0.22000	0.22474	0.15790	0.21981	0.000	0.004	0.000
0.15001	80	300	0.30000	0.30415	0.23507	0.29088	0.000	0.004	0.000
0.20001	68	368	0.36800	0.37436	0.31043	0.35478	0.000	0.003	0.000
0.25001	81	449	0.44900	0.43773	0.38359	0.41418	0.000	0.004	0.001
0.30001	54	503	0.50300	0.49565	0.45417	0.47047	0.000	0.002	0.001
0.35001	56	559	0.55900	0.54903	0.52183	0.52443	0.000	0.001	0.001
0.40001	47	606	0.60600	0.59853	0.58623	0.57650	0.000	0.000	0.001
0.45001	46	652	0.65200	0.64464	0.64707	0.62692	0.000	0.000	0.001
0.50001	41	693	0.69300	0.68773	0.70403	0.67579	0.000	0.000	0.000
0.55001	35	728	0.72800	0.72811	0.75684	0.72305	0.000	0.001	0.000
0.60001	43	771	0.77100	0.76602	0.80519	0.76856	0.000	0.001	0.000
0.65001	28	799	0.79900	0.80167	0.84881	0.81204	0.000	0.002	0.000
0.70001	23	822	0.82200	0.83524	0.88745	0.85308	0.000	0.004	0.001
0.75001	31	853	0.85300	0.86687	0.92082	0.89113	0.000	0.005	0.001
0.80001	30	883	0.88300	0.89670	0.94868	0.92542	0.000	0.004	0.002
0.85001	33	916	0.91600	0.92483	0.97078	0.95495	0.000	0.003	0.002
0.90001	36	952	0.95200	0.95137	0.98685	0.97843	0.000	0.001	0.001
0.95001	26	978	0.97800	0.97640	0.99667	0.99417	0.000	0.000	0.000
1.00000	22	1,000	1.00000	1.00000	1.00000	1.00000	0.001	0.045	0.012

$Min\int \left[K_{Copula_n}(t) - K_n(t) \right]^2 dK_n(t)$ **=** 0.001 for Gumbel, 0.045 for Clayton and 0.12 for Frank.

Below we show two VBA codes necessary for performing the nonparametric estimation that allows us to answer the question w*hich is the copula right one?*

VBA Code that computes Kendall using Tau nonparametric estimation

---

Function K_tau(ByVal X1 As Range, ByVal X2 As Range) As Double
'Estimate Kendall's correlation coefficient using the usual nonparametric estimate

$$\tau_n = \binom{n}{2}^{-1} \sum_{i<j} sign\left[\left(X_{1i} - X_{1j} \right) \left(X_{2i} - X_{2j} \right) \right] \qquad (1.32)$$

Dim i As Long, j As Long, s As Long, n As Long

n = X1.Rows.Count

For i = 1 To n
For j = i To n

```
        If j > i Then
        s = s + Sgn((X1.Cells(i, 1) - X1.Cells(j, 1)) * (X2.Cells(i, 1) - X2.Cells(j, 1)))
        End If
    Next
Next
K_tau = (Application.WorksheetFunction.Combin(n, 2) ^ -1) * s
End Function
```

VBA Code that computes the pseudo-observations $T_i = \{\text{number of } (X_{1j} < X_{1i})$ ***such that*** $X_{1j} < X_{1i}$ ***and*** $X_{2j} < X_{2i}\}$ ***/*** $(n-1)$ ***for*** $i = 1, 2, ..., n$.

```
Function Ts(ByVal X1 As Range, ByVal X2 As Range, i As Long) As Double

Dim j As Long, s As Long, n As Long
n = X1.Rows.Count
    For j = 1 To n
        If X1.Cells(j, 1) < X1.Cells(i, 1) And X2.Cells(j, 1) < X2.Cells(i, 1) Then
            Ts = Ts + 1
        End If
    Next
Ts = Ts / (n - 1)
End Function
```

Numerical Example

This example shows how value a first-to-default swap. For doing this, we use Li model (Li (2000)). Under this model, defaults are assumed to occur for individual assets according to Poisson process with a deterministic intensity called hazard rate h. This means that default times (T) are exponentially distributed with mean equal to $\frac{1}{h}$. Li relates the default times using a Gaussian (Normal) copula, we employ Gumbel copula, too.

We assume that:

- we have a portfolio of two credits $(n = 2)$.
- the contract is a two-year transaction $(t = 2)$, which pays one dollar if the first default happens during the first two years.
- Each credit has a, constant for term of de contract, hazard rate of $h = 0.10$.
- A constant interest rate of $r = 0.10$

The Pricing First-to-default Algorithm:

For each Monte Carlo trial we do the following:

- Draw uniform bivariates from chosen copula (Gaussian, Gumbel etc.)
- Map uniform to default times (T) using the inverse cumulative exponential distribution function given a fixed h.
- Compute minimum default time. If it is less than n, the present value of the contract is $1.e^{-r.T}$.

Then we average many trials and compute the expected value of the contract.

We examine the impact of the asset correlation on the value of the credit derivative using independence, perfectly correlated and using the following lineal correlation matrix:

Lineal Correlation	
1.000	0.629
0.629	1.000

Our simulation of 30,000 trials produces the following results:

1st-at-Default Swap

	Price
Independence	0.302
Perfectly Correlated	0.165
Normal Copula	0.249
Gumbel Copula	0.255

When we assume independence or perfect correlation below analytical solution is possible:

$$\frac{h_T}{r+h_T}\left(1-e^{-t(r+h_T)}\right) \quad (1.33)$$

where, in the independence case:

$$h_T = h.n \quad (1.34)$$

in the perfectly correlated case:

$$h_T = h \quad (1.35)$$

The result of the analytical solution is following presented:

Analytical Solution

	Price
Independence	0.301
Perfectly Correlated	0.165

Conclusion

There is clear evidence that equity returns have unconditional fat tails, to wit, the extreme events are more probable than anticipated by normal distribution, not only in marginal but also in higher dimensions. This is important both for market risk models as credit risk one, where equity returns are used as a proxy for asset returns that follow a multivariate normal distribution, and, therefore, default times have a multivariate normal dependence structure as well. Other than normal distribution should be used both in marginal as joint distributions. To overcome these pitfalls, the concept of copula, its basic properties and a special class of copula called Archimedean are introduced. Then we expose a guide to choose both the margins and the Archimedean copula that better fit to data. In addition, we provide an algorithm to simulate random bivariate from Archimedean copula. In order to cover the gap between the theory and its practical implementation VBA codes are provided. Finally we show a numerical example that illustrates the use of the copula by pricing a first-to-default contract.

For simplicity's sake, and given that the joint distribution is the major topic of this paper, when we value a first-to-default contract, we obviate in marginal distribution, to use a different distribution to normal one, but we employ Archimedean copula to model dependence structure.

This paper is accompanied by two spreadsheets that present step by step all the practical applications covered.

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Appendix A – Others Copula´s Parameters

Clayton Copula's Parameters

B.(*)	C.	D.	E.	F.	G.(**)
$\theta = \frac{2\tau}{1-\tau}$ (1.36)	$t^{-\theta} - 1$ (1.37)	$-\theta . t^{-\theta-1}$ (1.38)	$(1+t)^{-\frac{1}{\theta}}$ (1.39)	$t - \frac{(t^{\theta+1} - t)}{\theta}$ (1.40)	$-\frac{t^{\theta}(\theta+1)}{\theta} + \frac{1}{\theta} + 1$ (1.41)

(*) $\theta > 0$. Only positive dependence.

(**) There is not a closed form for the inverse distribution function $K^{-1}_{Clayton}$, so G. will be used for obtaining it by numerical root finding.

Frank Copula's Parameters

C.	D.	E.	F.	G.(**)
$-\ln \frac{e^{-\theta t} - 1}{e^{-\theta} - 1}$ (1.42)	$\frac{\theta}{1 - e^{\theta t}}$ (1.43)	$-\frac{\ln(e^{-\theta - t} - e^{-t} + 1)}{\theta}$ (1.44)	$t - \frac{\ln \frac{e^{-\theta t} - 1}{e^{-\theta} - 1}}{\theta}(e^{\theta t} - 1)$ (1.45)	$-e^{\theta t}\left[\ln\left(\frac{e^{\theta t} - 1}{e^{\theta} - 1}\right) + \theta(1-t)\right]$ (1.46)

(*) $-\infty < \theta < \infty$. Positive and negative dependence.

(**) There is not a closed form for the inverse distribution function K^{-1}_{Frank} , so G. will be used for obtaining it by numerical root finding.

B. Frank copula has not close form that allows us to calculate theta parameter. We use numerical root finding for calculating it. Press **tau Frank** button in the attached Excel sheet for performing the following VBA code:

```
Private Sub CommandButton1_Click()

Dim ktau As Double
' Input wanted Kendall Tau τ
 ktau = InputBox("Kendal Tau: ", "Input")

' In ChangingCell "G4" we use formula (1.50) See Appendix B
Range("G3").GoalSeek Goal:=ktau, ChangingCell:=Range("G4")

End Sub
```

Appendix B – Kendall τ revisited

The Kendall Tau can be calculated both using formula (1.32) or the following one:

Let X and Y be random variables with an Archimedean copula C generated by φ, Kendall's Tau of X and Y is given by:

$$\tau_\theta = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt \tag{1.47}$$

when C is Gumbel (1.47) is given by:

$$\tau_\theta = 1 + 4\int_0^1 \frac{(-\ln t)^\theta}{-\theta(\ln t)^{\theta-1}\frac{1}{t}}dt = 1 + 4\int_0^1 \frac{t\ln t}{\theta}dt = 1 - \frac{1}{\theta} \tag{1.48}$$

when C is Clayton (1.47) is given by:

$$\tau_\theta = 1 + 4\int_0^1 \frac{t^{-\theta} - 1}{-\theta.t^{-\theta-1}}dt = 1 + 4\int_0^1 \frac{t^{\theta+1} - t}{\theta}dt = \frac{\theta}{\theta+2} \tag{1.49}$$

when C is Frank (1.47) is given by:

$$\tau_\theta = 1 - \frac{4\left(1 - D_1(\theta)\right)}{\theta} \tag{1.50}$$

where $D_k(x)$ is the Debye function, given by:

$$D_k(x) = \frac{k}{x^k}\int_0^x \frac{t^k}{e^t - 1}dt \tag{1.51}$$

We use (1.50) for calculating Frank copula's τ. The integral $\left(\int_a^b f(x)dx\right)$ is calculated using Riemann sums method. It approximates the integral by dividing the interval $[a,b]$ into m subintervals and approximating f with a constant function on each subinterval. Riemann sum approximates our definite integral with:

$$\int_a^b f(x)dx \approx \sum_{k=1}^m f(x^k)\Delta x \tag{1.52}$$ [15]

15 We use a VBA code to figure out this integral. For its right working, it is necessary to contain Microsoft Script Control 1.0 as a reference in VBA module. If Microsoft Script Control 1.0 is missing you could download from **http://www.microsoft.com/downloads/details.aspx?FamilyId=D7E31492-2595-49E6-8C02-1426FEC693AC&displaylang=en**

Appendix C - Using Simtools features in VBA programs.

For performing Monte Carlo Simulation we use a freeware called Simtools. For its right working, it is necessary to attach Simtools.xla as a reference in VBA module, by applying the Tools: **References** menu command in the Visual Basic Editor and checking Simtools.xla as an available reference. More information about Simtools click **here** .

Appendix D – Tail dependence.

An example can be useful to visualize the issue.

We assume:

- Kendall Tau τ = 0.45835.
- ρ = 0.65937. [16]
- Gumbel copula (1.12) and Gaussian one[17]
- Use (1.19) to calculate θ.

Upper Tail Dependence								
Gumbel Copula								
τ	**0.45835**							
θ	**1.84623**							
$u \to 1$	0.99000	0.99250	0.99500	0.99750	0.99900	0.99950	0.99990	0.99995
$C_{Gumbel}(u,u)$	0.98548	0.98910	0.99273	0.99636	0.99854	0.99927	0.99985	0.99993
λ_{upper} (1.5)	***0.548***	***0.547***	***0.546***	***0.545***	***0.545***	***0.545***	***0.544***	***0.544***
Gaussian Copula								
ρ	**0.65937**							
$u \to 1$	0.99000	0.99250	0.99500	0.99750	0.99900	0.99950	0.99990	0.99995
$C_{Gaussian}(u,u)$	0.98232	0.98661	0.99097	0.99541	0.99813	0.99905	0.99981	0.99990
λ_{upper} (1.5)	***0.232***	***0.215***	***0.194***	***0.163***	***0.129***	***0.109***	***0.074***	***0.063***

Lower Tail Dependence	
Gumbel Copula	
τ	**0.45835**
θ	**1.84623**

16 We use the following relationship: $\rho_{ij} = \sin\left(\frac{\pi}{2}\tau_{ij}\right)$, $\tau_{ij} = \frac{2}{\pi}\arcsin\left(\rho_{ij}\right)$

17 $C_{Gaussian}(u,v;\rho) = \Phi\left(\Phi^{-1}(u),\Phi^{-1}(v)\right)$, where Φ denotes the joint distribution function of the bivariate standard normal distribution function with linear correlation ρ , and Φ^{-1} denotes the inverse of the distribution function of the univariate standard normal distribution. In Excel language $\Phi = \text{NormSInv}(u)$. For Φ^{-1} a VBA code is available **here**.

$u \to 0$	0.01000	0.00750	0.00500	0.00250	0.00100	0.00050	0.00010	0.00005
$C_{Gumbel}(u,u)$	0.00123	0.00081	0.00045	0.00016	0.00004	0.00002	0.00000	0.00000
λ_{lower} (1.8)	***0.123***	***0.108***	***0.089***	***0.065***	***0.043***	***0.031***	***0.015***	***0.011***
Gaussian Copula								
ρ	**0.65937**							
$u \to 0$	0.01000	0.00750	0.00500	0.00250	0.00100	0.00050	0.00010	0.00005
$C_{Gaussian}(u,u)$	0.00232	0.00161	0.00097	0.00041	0.00013	0.00005	0.00001	0.00000
λ_{lower} (1.8)	***0.232***	***0.215***	***0.194***	***0.163***	***0.129***	***0.109***	***0.074***	***0.063***

In the Gumbel copula's case when $u \to 1$ the tail upper dependence changes slightly. In Gaussian copula's case the upper tail dependence tends to zero.

When $u \to 0$ the lower tail dependence tends to zero for Gaussian copula and Gumbel one. So that, our example suggests that Gumbel copula has upper tail dependence but does not has lower tail one, whereas Gaussian copula does has neither.

The formulae for calculating the upper tail dependence from Gumbel copula is:

$$2 - 2^{\left(\frac{1}{\theta}\right)} \tag{1.53}$$

in this case:

$$2 - 2^{\left(\frac{1}{1.84623}\right)} = 0.5444 \tag{1.54}$$