Interest Rate Derivatives: An Introduction to the Pricing of Caps and Floors

Abukar M Ali and Mohamoud B Dualeh

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Abstract
This article introduces the basic structure and engineering of interest rate derivative instruments, which are products whose payoffs depend in some way on the level of interest rates. These financial instruments include caps, floors, swaptions and options on coupon-paying bonds. The most common way to price interest rate derivatives such as caps and floors, is to adopt the Black-Scholes approach and to implement the Black (1976) pricing model. Following an introduction to the structure of interest rate derivatives, we also present the underlying risk neutral representation of the Black model in order to derive the existing closed form solution. In fact, the model is very intuitive and easy to implement for a single cap/floor. When pricing a portfolio of caplets/floorlets however, with multiple expiry dates, one may need to use sophisticated analytics written in higher programming languages for computational speed and efficiency.

Interest-rate caps and floors
Interest rate options are widely used to either speculate on the future course of interest rates or to hedge the interest payments or receipts on an underlying position. The advantage of these instruments over other types of derivatives such as swaps and interest rate futures is that interest options allow an investor to benefit from changes in interest rates while also limiting any downside losses. Hence, like all options they provide insurance.

The over-the-counter market trades options on a number of interest rates relating to short-term financial instruments, such as bank deposits, certificates of deposit, commercial paper, and T-bills. The most liquid options traded of all these are caps, floors, and collars. Caps are interest rate option structures with a payout if interest rates rise (this may also depend on the option style or exercise). Consequently, they are used by floating rate borrowers or issuers to ensure against a rise in interest rates. Floors, on the hand, have a payoff for the user if interest rates fall and, consequently, are used by depositors/investors to insure against interest rates falling. A collar is a combination of the two while a zero cost collar can be constructed by taking opposite position of the two options types, such that the strike prices are chosen so that the net premium for the user is zero.

Consider the following example where a corporation has issued a floating-rate note or a loan, paying interest semi-annually at six-month Libor + 0.50%, with residual term to maturity of 5 years and 3 months. You can effectively lock in the maximum level of your future borrowing rate by buying a cap that consists of 10 half-year caplets, starting in 3 month's time.

Generic representation of the payoff to a cap is given in Figure 1.1. The long option position is a call if the underlying is an interest rate or an FRA; it is a put if the underlying instrument is a future. The long-option position combined with the unexpected effects of interest rate changes gives a payoff that resembles a long put position on the interest rate – i.e. if interest rates fall the borrower benefits from the fall less the option premium, but if rates rise the interest rate payable on is capped. The effective rate of interest on the capped loan will be the exercise price of the option plus the option premium and plus (minus) any reset margin above (below) Libor. It is in fact very intuitive to see the payoff to the cap from the graph. The solid thin line represents the interest rate exposure to the issuer or a borrower of a floating rate loan. The thin dotted line represents the long call option for hedging the loan. The payoff of the combined position is represented by the solid fat line, which shows the offsetting effect of the positions for various interest rate scenarios.

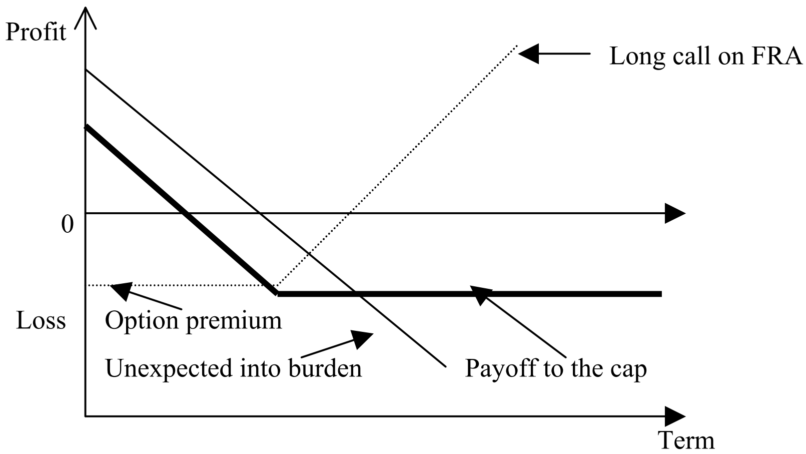

Figure 1.1 Profit and Loss using a Long Call on FRA

Caps and Floors payoffs

The pay off of the options can simply be described algebraically using the following notation. Let K_c and K_p be the strike levels for the call and the put respectively while T is the expiration date of the contract.

Caplet (pay off at maturity) $$Q\{\max,(0, Libor_T - K_c)\frac{days}{360}\}$$

Floorlet (pay off at maturity) $$Q\{\max,(0, K_P - libor_T)\frac{days}{360}\}$$

The implication from the above algebrical representation of the payoffs at settlement requires knowledge of the future course of the underlying rates, i.e. LIBOR. Since we cannot realistically forecast the future course of an interest rate, it is natural to model it as a random variable. Now, following the Black-Scholes[1] framework, the basic assumption underlying option pricing theory is the nonexistent of arbitrage, were the word "arbitrage" essentially addresses the opportunity to make a risk-free profit. In other words, the common saying that "there is no free lunch" is the fundamental principal underlying the theory of finance. To make the ideas presented above more concrete, we now begin a formal treatment of the stochastic process of interest rate.

Let us suppose that the interest rate r follows Brownian Motion described by a stochastic differential equation of the form

[1] Black, F & Scholes, M 1973 “The pricing of options and corporate liabilities” *Journal of Political Economy* 81 (1973), 637-659

$$dr_t = u(t,r_t)dt + \sigma(t,r_t)dW_t \quad (1)$$

where $u(t,r_t)$ and $\sigma(t,r_t)$ the expected value and the standard deviation of the instantaneous interest rate variation, respectively. The price at date *t* of a zero-coupon bond maturing at date $T > t$ is a function of the short term interest rate

$$B(t,T) = B(t,T,r). \quad (2)$$

Finally, the prices of zero-coupon bonds are derived by using an approach based on a parabolic partial differential equation (PDE). From the PDE approach and applying Ito's lemma[2] to equation (2) and using

$$dr_t = u(t,r_t)dt + \sigma(t,r_t)dW_t$$

one gets

$$dB = [\frac{\delta B}{\delta t} + \frac{\delta B}{\delta r}u + \tfrac{1}{2}\sigma^2 \frac{\delta^2 B}{\delta r^2}]dt + \frac{\delta B}{\delta r}\sigma dW_t \quad (3)$$

$$= u_B B dt + \sigma B dW_t. \quad (4)$$

We now set up a riskless portfolio: $P = B_1 + \Phi B_2$ involving two zero-coupon bonds, described as

$$B_1 = B(t,T_1,r) \quad (5)$$
$$B_2 = B(t,T_2,r). \quad (6)$$

Selecting the position Φ in the second bond which renders the portfolio to be riskless, we find the following condition:

$$\frac{\delta B_1}{\delta r} + \Phi \frac{\delta B_2}{\delta r} = 0. \quad (7)$$

The net effect of the change of the portfolio is equal to zero. We can determine the appropriate value of the position in the second bond by simply rearranging the equation (7), from which we get

$$\Phi = -\frac{\delta B_1}{\delta r} \Big/ \frac{\delta B_2}{\delta r}. \quad (8)$$

This can also be interpreted as the ratio of the sensitivities to the risk variable[3]. We can now express the change of the portfolio in a time *dt* as

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[2] See John Hull, *Options, Futures and Other Derivatives*, 4th edition, McGraw-Hill 2000. See also the Appendix of this article.
[3] Note, this is quite similar to the idea of hedging a bond portfolio

$$d\Pi = [\frac{\delta B_1}{\delta t} + \frac{\delta B_1}{\delta r}u + \tfrac{1}{2}\sigma^2 \frac{\delta^2 B_1}{\delta r^2}]dt + \Phi[\frac{\delta B_2}{\delta t} + \frac{\delta B_2}{\delta r}u + \tfrac{1}{2}\sigma^2 \frac{\delta^2 B_2}{\delta r^2}]dt \qquad (9)$$

Now, since the portfolio is riskless, it should have a return equal to the risk-free rate.

$$\frac{d\Pi}{\Pi} = rdt$$

or equivalently[4]

$$d\Pi = [\frac{\delta B_1}{\delta t} + \frac{\delta B_1}{\delta r}u + \tfrac{1}{2}\sigma^2 \frac{\delta^2 B_1}{\delta r^2}]dt + \Phi[\frac{\delta B_2}{\delta t} + \frac{\delta B_2}{\delta r}u + \tfrac{1}{2}\sigma^2 \frac{\delta^2 B_2}{\delta r^2}]dt = r(B_1 + \Phi B_2)$$

The randomness in $d\Pi$ will vanish since it makes the coefficient of dr zero. If we assume the non existence of arbitrage, the portfolio must have a rate of return equal to the short rate of interest. Finally, using equation (8) and rearranging the above equation, we have

$$\frac{\frac{\delta B_1}{\delta t} + \frac{\delta B_1}{\delta r}u + \tfrac{1}{2}\sigma^2 \frac{\delta^2 B_1}{\delta r^2} - rB_1}{\frac{\delta B_1}{\delta r}} = \frac{\frac{\delta B_2}{\delta t} + \frac{\delta B_2}{\delta r}u + \tfrac{1}{2}\sigma^2 \frac{\delta^2 B_2}{\delta r^2} - rB_2}{\frac{\delta B_2}{\delta r}} \qquad (10)$$

Hence the quantity (the same as the above equation except that we have divided by σ)

$$\lambda = \lambda(t, r_t) = \frac{\frac{\delta B}{\delta t} + \frac{\delta B}{\delta r}u + \tfrac{1}{2}\sigma^2 \frac{\delta^2 B}{\delta r^2} - rB}{\frac{\delta B}{\delta r}\sigma} \qquad (11)$$

is a constant across all obligations at a given date. That is to say, the price of risk is constant. Dividing the numerator and denominator by B in equation (11), we get

$$\lambda = \frac{u_B - r}{\frac{1}{B}\frac{\delta B}{\delta r}\sigma}$$

or equivalently

$$u_B - r = \lambda \sigma_B \qquad (12)$$

Equation (12) can be interpreted as the access return (the return on a bond above the risk-free rate) is equal to λ times a factor measuring the risk (volatility) of the bond. Hence, λ can be interpreted naturally as the market price (interest rate) of risk.

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[4] For lack of space, we follow the equation numbering and write it here as 2.0

It follows that the values of any securities that are sensitive to the change of the interest rate *r* satisfies the partial-differential equation:

$$\frac{\delta B}{\delta t}+\frac{\delta B}{\delta r}u+\tfrac{1}{2}\sigma^2\frac{\delta^2 B}{\delta r^2}-rB=\frac{\delta B}{\delta r}\sigma\lambda \qquad (13)$$

or

$$\frac{\delta B}{\delta t}+\frac{\delta B}{\delta r}(u-\lambda\sigma)+\tfrac{1}{2}\sigma^2\frac{\delta^2 B}{\delta r^2}-rB=0 \qquad (14)$$

The stochastic model for the spot rate presented above allows us to value contingent claims such as bond options. In our analysis we can price caps and floor by solving equation (14) with the boundary condition $B(T,T,r)=1$.

The result from equation (14) is a modified version of the original Black-Scholes solution for pricing derivatives via risk neutral technique. This solution is the most elegant result for pricing derivatives as it provides neat and incredible mathematical solution to a complicated issue that incorporates investor risk. In fact one of the crucial aspect of the model is based on its assumption on complete markets. This implies that that all derivatives can be priced via replication strategy and it further assumes that the appropriate securities to create the replicate position always exist. Under this scenario, the return from the hedged or the replicated position should yield the risk free rate and the risk premium required by investors is solved via the above partial-differential equation. In fact, you can see this by rearranging the equation and taking the -*rB* to the right hand side and equate this to the change of the portfolio.

Having derived the general bond pricing and risk-neutral framework, one can solve equation (14) while using the boundary conditions to arrive the Black-76 closed-from solution to price interest options such caps/floor, and bond option[5]. For caps and floors, we will use the implied forward rate *F,* at each caplet maturity as the underlying asset. We assume also that underlying forward rates follow lognormal process. The price of the cap is simply the sum of the price of the caplets that make up the cap. Similarly, the value of a floor is the sum of the sequence of individual put options, these are often called floorlets.

$$Cap=\sum_{i=1}^{n}Caplet_i \qquad\qquad Floor=\sum_{i=1}^{n}Floorlet_i$$

where

$$Caplet=\frac{Notional\dfrac{d}{Basis}}{(1+F\dfrac{d}{Basis})}e^{-r(T-t)}[FN(d_1)-XN(d_2)]$$

[5] For proof of the closed-from solution see Paul Wilmott, *Derivatives- The theory and practice of financial engineering*, J Wiley & Sons 1999.

$$Floorlet = \frac{Notional \frac{d}{Basis}}{(1 + F\frac{d}{Basis})} e^{-r(T-t)}[XN(-d_2) - FN(-d_1)]$$

and

$$d_1 = \frac{\ln(F/X) + (\sigma^2/2(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

The term *d* is the number of days in the forward period . *Basis* is the number of days in the year used in the market, and the term *N(.)* is the cumulative normal distribution. Note that the risk-free interest rate does not enter the equation for d_1 and d_2 because the influence of the risk-free rate on the future values of the underlying asset in a risk-neutral world is already accounted for in the forward price. Now, you can easily implement this model by simply plugging in the input parameters such as, a value for the underlying asset, strike level, number of days in the forward period and year. Most importantly, you will need to assume a value for the volatility parameter. Once these values are entered, the value of a cap or floor can easily be determined.

To implement the model described above and generate a numerical values, we will use the Bloomberg Professional Analytics as our pricing tool of choice. The Bloomberg cap/floor/collar pricing capabilities has become market standard among various market players such as derivative traders, sales, and risk managers. Use of these analytics creates transparency and helps market practitioners assess risk and execute trades very easily. To continue our discussion, we analyse 5 year cap on a 3 month LIBOR. Figure 1.2. displays the value of the caplets expressed as a percentage of face amount as well as the market value in nominal terms.

We illustrate the form for pricing the caplet on the Bloomberg system. This is done by selecting the function BCCF <go> and then entering the following parameters: settlement date, start date and expiration date and the face amount of the contract. Volatility and strike(s) are crucial parameters to the model and will have tremendous effect on the cost of the option depending on whether you wish to price the option on a flat or varying strike and volatility levels. Enter a single volatility and strike for all maturities or simply page forward to the second page and enter unique strike and volatility levels for each caplet components. Once you have entered the strike(s), you will immediately see the intrinsic value of the option visually from the graph on the first page. The red horizontal line and the white steep curve display the strike level and the implied forward[6] rate respectively.

[6] The implied forward rate is determined using risk-neutral technique from the dollar swap curve and can be accessed from the Bloomberg Forward Curves by typing: FWCV <go>.

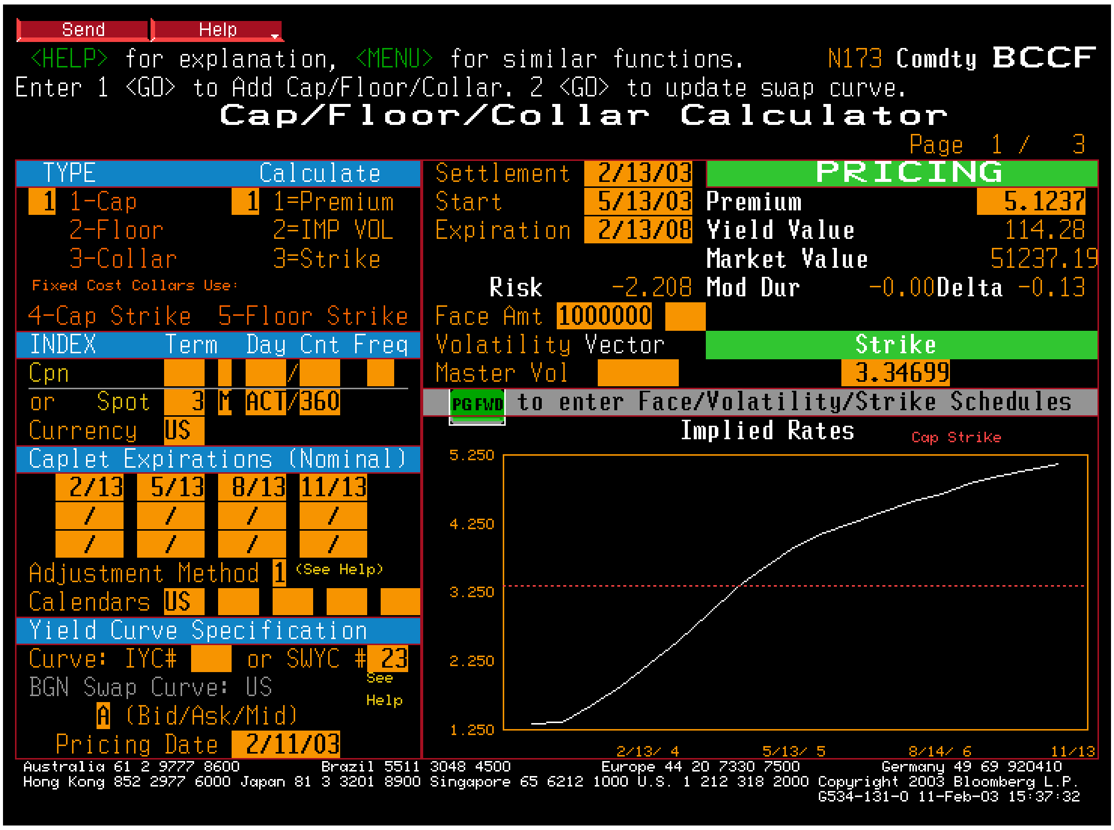


Figure 1.2. Bloomberg Cap/Floor/Collar Valuation Screen: BCCF <go>

The selected flat option strike level of 3.501 is in fact higher then the implied forward rate until approximately 12/27/04 but it is lower after this date until the expiry of the last option. So, from the point of view of intrinsic value, the option is out-of-the money until December 2004 and in-the-money from March 2005 until the last expiration date. We use the models default cap/floor implied volatilities and get an option premium of:

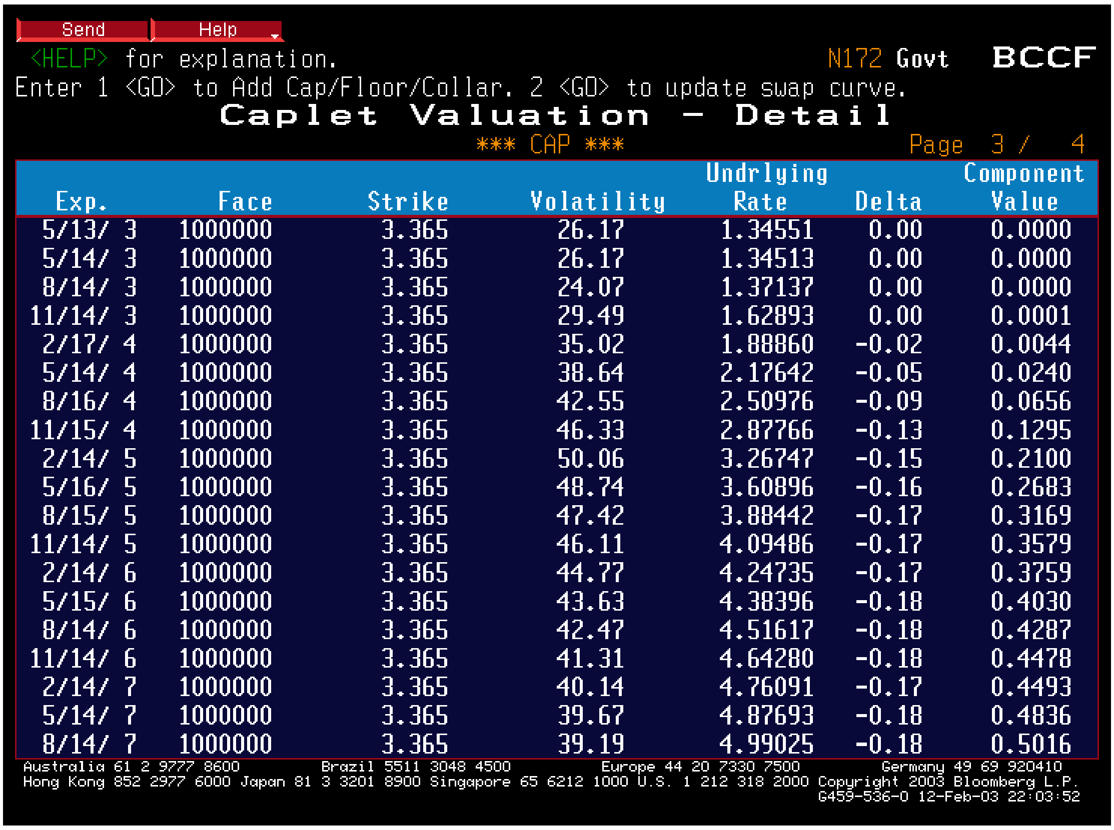
$$Cap = \sum_{i=1}^{n} Caplet_i = 5.0781\%$$

and corresponding market value of

$$(\frac{5.0781}{100})x1,000,000 = 50781.00$$

To see the impact of the cost of the option by simply changing a single parameter input such as the strike level, increase the strike by for example one percentage point and you will price the option cheaper. To demonstrate the computational speed of the model, change the calculator option to solve for the implied volatility by entering specific premium level. Through iterative process the model will work out the correct immediately implied volatility given an option premium.

Page three of the Bloomberg Cap/Floor/Collar pricing screen BCCF provides detailed information such as each expiry dates of the option components, volatilities, implied forward rates, deltas, and the option component values. This is shown at figure 1.3.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<HELP> for explanation. N172 Govt BCCF
Enter 1 <GO> to Add Cap/Floor/Collar. 2 <GO> to update swap curve.
Caplet Valuation – Detail
*** CAP *** Page 3 / 4

Exp.	Face	Strike	Volatility	Undrlying Rate	Delta	Component Value
5/13/ 3	1000000	3.365	26.17	1.34551	0.00	0.0000
5/14/ 3	1000000	3.365	26.17	1.34513	0.00	0.0000
8/14/ 3	1000000	3.365	24.07	1.37137	0.00	0.0000
11/14/ 3	1000000	3.365	29.49	1.62893	0.00	0.0001
2/17/ 4	1000000	3.365	35.02	1.88860	-0.02	0.0044
5/14/ 4	1000000	3.365	38.64	2.17642	-0.05	0.0240
8/16/ 4	1000000	3.365	42.55	2.50976	-0.09	0.0656
11/15/ 4	1000000	3.365	46.33	2.87766	-0.13	0.1295
2/14/ 5	1000000	3.365	50.06	3.26747	-0.15	0.2100
5/16/ 5	1000000	3.365	48.74	3.60896	-0.16	0.2683
8/15/ 5	1000000	3.365	47.42	3.88442	-0.17	0.3169
11/14/ 5	1000000	3.365	46.11	4.09486	-0.17	0.3579
2/14/ 6	1000000	3.365	44.77	4.24735	-0.17	0.3759
5/15/ 6	1000000	3.365	43.63	4.38396	-0.18	0.4030
8/14/ 6	1000000	3.365	42.47	4.51617	-0.18	0.4287
11/14/ 6	1000000	3.365	41.31	4.64280	-0.18	0.4478
2/14/ 7	1000000	3.365	40.14	4.76091	-0.17	0.4493
5/14/ 7	1000000	3.365	39.67	4.87693	-0.18	0.4836
8/14/ 7	1000000	3.365	39.19	4.99025	-0.18	0.5016

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Figure 1.3. Bloomberg Cap/Floor/Collar Valuation Screen: BCCF <go>

Now, once the premium is computed, it must also be discounted back to time zero using the forward rate consistent to the expiry of the contract. Bloomberg's cap/floor/collar analytics is a powerful tool to price these contracts and requires minimum model required inputs as it will also compute the implied forward rates applicable for different expiry dates for the option.

Estimating volatility

The estimation of volatility is of course central to the pricing options. Specifically, the volatility of returns from today until the maturity of the option is required. The black model assumes constant volatility over this period, unlike the stochastic volatility and some of the numerical models. The most common approaches in estimating volatilities is to adopt either of the following techniques:

- To assume volatility is stationary, and to calculate historical volatility

- To model volatility based upon historic information to provide a forecast using models such as ARCH (Autoregressive Conditional Heteroskedasticity)[7]
- To imply volatility from other options already trading in the market place

The latter approach suggests a circular argument, namely deriving the volatility from existing options. It can however act as an extremely useful check on where other participants see volatility, but must be interpreted carefully.

This is precisely what is referred to in most academic literatures as the market expectation of future rate changes. Many option markets that are highly liquid, for example at-the-money USD and GBP cap markets, will quote volatilities rather than option prices. This is because all the pricing parameters required for the black model are available elsewhere such as in the swap market. Therefore volatility is the only unknown parameter.

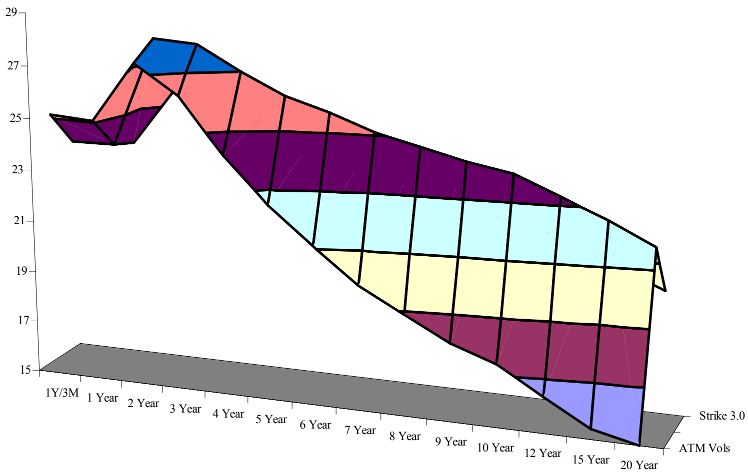

Figure 1.4. Bloomberg Cap/Floor Implied volatility surface

An excellent knowledge of implied volatility in terms of future expectations for future price or rate variability allows one a better way to assess his/her exposure to market risk. Knowing how changing volatility affect hedging and pricing will be crucial for the development of a cost-effective hedges and weigh up the timing risks coupled with a particular strategy.

It is also a common practise to use volatility surfaces, i.e. a matrix of (strike vs. forward start date), when pricing and valuing caps and floors. This allows the smile effect to be incorporated. The data used in figure 1.4 can be accessed or easily downloaded from the Bloomberg in order to generate three dimensional volatility surfaces and smiles to help one identify miss-priced options.

[7] Tsay Ruey S, *Analysis of Financial time series*, Wiley, 2002

Appendix: Itô's Lemma

In finance, when using continues-time models, it is common to assume that the price of an asset is an Ito's process. Therefore, to derive the price of a financial derivative, one needs to use Ito's calculus. In this section, we briefly review Ito's lemma by treating it as a natural extension of the differentiation in calculus. Ito's lemma is the basis of stochastic calculus.

Review of Differentiation

Let *G(x)* be differentiable function of *x*. Using Taylor expansion, we have

$$\Delta G \equiv G(x+\Delta x) - G(x) = \frac{\delta G}{\delta x}\Delta x + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}(\Delta x)^2 + \frac{1}{6}\frac{\delta^3 G}{\delta x^3}(\Delta x)^3 + \dots\dots\dots$$

Taking the limit as $\Delta x \to 0$ and ignoring the higher order terms of, we have

$$dG = \frac{\delta G}{\delta x}dx.$$

When *G* is a function of *x* and *y,* we have

$$\Delta G = \frac{\delta G}{\delta x}\Delta x + \frac{\delta G}{\delta y}\Delta y + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}(\Delta x)^2 + \frac{\delta^2 G}{\delta x \delta y}\Delta x \Delta y + \frac{1}{2}\frac{\delta^2 G}{\delta y^2}(\Delta y)^2 + \dots\dots\dots$$

Taking the limit as $\Delta x \to 0$ and as $\Delta y \to 0$, we have

$$dG = \frac{\delta G}{\delta x}dx + \frac{\delta G}{\delta y}dy.$$

To turn in the case in which G is a differentiable function of x_t and *t,* and x_t is an Ito's process. The Taylor expansion becomes

$$\Delta G = \frac{\delta G}{\delta x}\Delta x + \frac{\delta G}{\delta t}\Delta t + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}(\Delta x)^2 + \frac{\delta^2 G}{\delta x \delta t}\Delta x \Delta t + \frac{1}{2}\frac{\delta^2 G}{\delta t^2}(\Delta t)^2 + \dots\dots\dots$$

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