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Credit Risk Models and the Valuation of Credit Default Swap Contracts

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Abstract
This paper surveys some of the main credit risk models within structural models and reduced-form models. In particular, it focuses on the Merton model and its extensions under the structural models. It also concentrates on intensity based models such as Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1997), and Duffie and Singleton (1998). Empirical results investigating the differences between market-quoted credit default swaps premium and model implied CDS premiums are presented. Finally, the Kettunen, Ksendzovsky, and Meissner (KKM) model (2003) is reviewed and implemented to compute credit default swap premium for a given set of data. From the existing research on credit risk models, reduced form models seems to be the preferred approach when pricing a firm's risky debt or related credit derivatives.

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Contents

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1. Introduction.

Pricing credit derivatives and credit risk in general, is quite similar in technique to pricing traditional derivatives, such as interest rate swaps or stock options. This paper focuses on the investigation on two general methods for valuing default risk claims and by extending these models to valuation of credit derivatives in particular default swap or credit default swap contracts (CDS). The models or approaches investigated are the structural and reduced form models. We will examine the suitability of these models to the pricing of credit protection in rapidly growing credit default swap market by identifying some of the key advantages and drawback.

The following are some of the key questions that this paper is concerned to investigate. How is credit default swap priced?. Which model is most appropriate model to use for pricing implementations?. We will use reduced or intensity based model to implement pricing default swaps using corporate bond yields and solve for the default swap premium they imply. We compare these implied credit default swap premium to actual market CDS prices. Implied premium tend to be much higher than then the CDS prices quoted in the market. What accounts for these differences?. The differences are related to measures of Treasury special-ness, corporate bond illiquidity, and coupon rates of the underlying bonds, suggesting the presence of important tax related and liquidity components in corporate spreads. Also, both credit derivatives and equity markets tend to lead the corporate bond market.

There are number of issues that may arise from the implementation when pricing the credit default swap, for example, what are the assumptions underlying the pricing model?. What are the implications for relaxing some of these assumptions? For example, we will assume no counter party default and that interest rate, default probabilities and recovery rates are independent. The one parameter necessary for valuing default swap that cannot be observed directly in the market is the expected recovery rate. Hull & White (2000) assume that the same recovery rate is used for estimating default probability densities and for calculating the payoff. As it happens there is an offset. As the expected recovery rate increases, estimates of the probability of default increase and payoff decrease[1]. The overall impact of the recovery rate assumption on the value of a credit default swap is generally fairly small when the expected recovery is in the 0% - 50% range.

[1] See Appendix C for default probability term structures under various recovery rates and credit ratings.

1.1. Structural Models.

Structural credit pricing models are based on modelling the stochastic evolution of the balance sheet of the issuer, with default when the issuer is unable to or unwillingly to meet its obligations. In this model the asset value of the firm is assumed to follow a diffusion process and default is modelled as the first time the firm's value hit a pre-specified boundary. Because of the continuity of the process used, the time of default is a predictable stopping time. The models of Merton (1974), Black and Cox (1976), Geske (1977) Longstaff and Schwartz 1993 and Das 1995 are representatives of this approach.

1.2 Reduced-Form Models/ Intensity Models

In the intensity models the time of default is modelled directly as the time of the first jump of a Poisson process with random intensity. The first models of this type were developed by Jarrow and Thurnbull (1995), Madal and Unal (1998) and Duffie and Singleton (1997). Jarrow and Thurnbull assume default is driven by a Poisson process with constant intensity and known payoff at default. Duffie and Singleton (1997) model assumes the payoff when default occurs as cash, but denoted as a fraction (1-*q*) of the value of defaultable security just before default. This model was applied to a variety of problems including swap credit risk, two sided credit risk and pricing credit default swap, binary credit default swap and credit default swap option. Table 1.1 provides brief overview on the existing credit risk model’s main advantages as well as their limitations.

Table 1.1
Strengths and Drawbacks of Various Models for Default Risky Bonds and Swaps

Model	**Advantages**	**Drawbacks**
Merton (1974)	Simple to implement	Requires inputs from value of the firm. Default occurs only at the maturity of the debt. Information in the history of defaults and credit rating changes cannot be used.
Longstaff and Schwartz (1995)	Simple to implement. Allows for stochastic term structure and correlation between defaults and interest rates	Requires inputs from value of the firm. Information in the history of defaults and credit rating changes cannot be used.

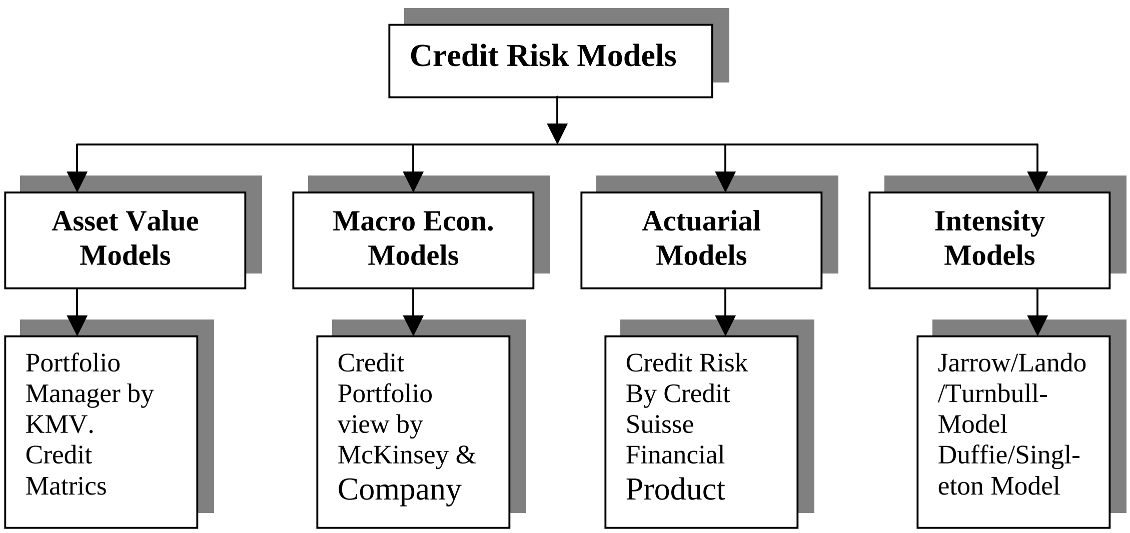
Jarrow, Lando, and Turnbull (1997)	Simple to implement. Can exactly match the existing prices of default-risky bonds to infer risk-neutral default probabilities of default and credit rating changes. Uses the information in the history of defaults and credit rating changes.	Correlation not allowed between default probabilities and the level of interest rates. Credit spreads changes only when credit rating change.
Lando (1998)	Allows correlation between default probabilities and interest rates. Allows many existing term structure models to be easily embedded in the valuation framework.	Historical default probabilities and credit rating changes are used under the assumption that the risk premiums due to defaults and rating changes are zero.

Model	**Advantages**	**Drawbacks**
Duffie and Singleton (1997)	Allows correlation between default probabilities and interest rates. Recovery can be random and depend on the pre-default value of the security A default free term structure model can be accommodated, and existing valuation results for default free term structure models can be readily used.	Historical credit changes and defaults cannot be used.
Duffie and Haug (1996) (swaps).	Has all the advantages of Duffie and Singleton.	Historical credit changes and defaults cannot be

	ISDA guidelines for settlement upon swap default can be incorporated.	used. Difficult for computational reasons especially for cross currency swaps if domestic and foreign interest rates are random or stochastic.

Figure 1.1 illustrates the types of credit risk models available and the focus of this paper is the implementation of the asset value models and the intensity based models.

Figure 1.1: Credit Risk Models



2.1 Structural Credit Risk Models.

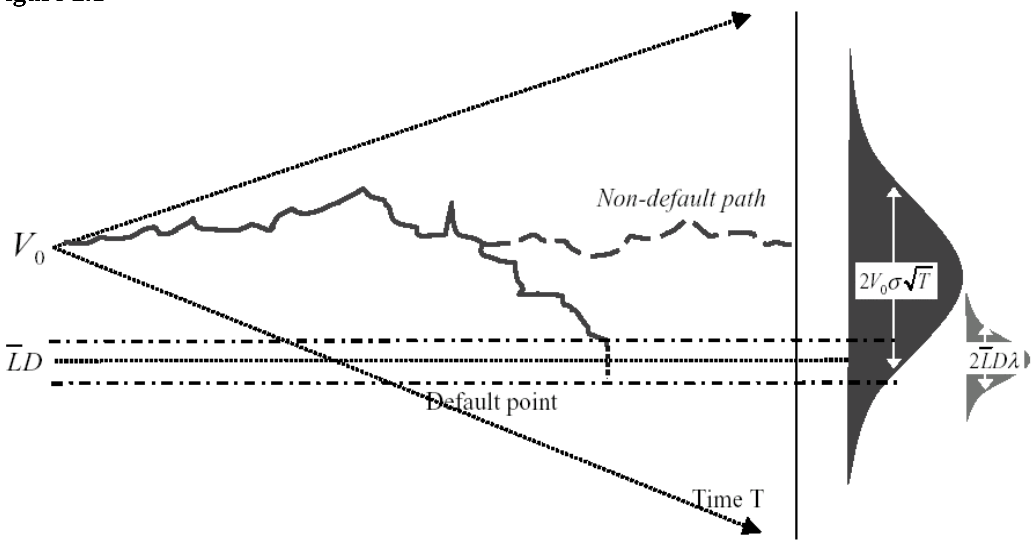
2.1.1 Merton's Model.

The basic foundations of structural models have been laid in the seminal paper Merton (1974). Here it is assumed that a firm is financed by equity and single zero-coupon bond with notional amount or face value of *K* and maturity *T*. The firm's contractual obligation is to repay the face value (*K)* of the debt to the bond investors at the maturity of the debt (*T).* In the event of default, bond holders will assume ownership of the firm. Hence the default time τ is a discrete random variable given by

$$\tau = \begin{cases} T & if \quad V_T < K \\ \infty & if \quad else. \end{cases}$$

Figure 2.1 shows the triggering of default as soon as the stochastic path of the firm value crosses the default barrier which is the face value of the debt at any time between time zero and *T*. This however is an extension of Merton's model which relaxes the assumption of default taking place only at maturity of the debt. Under Merton's model, default cannot occur prior to the maturity of the debt. This means that default is only triggered if the asset value exceeds the total outstanding debt of the firm at time *T*.

Figure 2.1



The dynamics of the firm value under the probability measure $I\!P$ follows a geometric Brownian motion:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t, \qquad V_0 > 0,$$
$$dV_t = \mu V_t dt + \sigma V_t dW_t, \qquad V_0 > 0, \qquad (1)$$

Where $\mu \varepsilon R$ is a drift parameter, $\sigma > 0,$ is a volatility parameter for the firm, and W_t is a standard Brownian motion. The solution for equation *(1)* is given by[2]:

$$V_t = V_0 \exp(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$$

We apply Ito's formula and the above stochastic differential equation (SDE) for the firm and formulate the Black & Scholes differential equation.
Ito's Lemma: let $F(V_t\ t)$ be twice-differentiable function of *t* and of the random process V_t given in *(1)* with well behaved drift and diffusion parameters, $\mu,$ and σ [3]. Then we have

[2] Joshi M.S. "The concepts and practice of mathematical finance" (2003). Cambridge University Press

$$dF_t = \frac{\partial F}{\partial V_t} dV_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial V^2_t} dV_t^2 \qquad (2)$$

Using Ito's multiplication rules, the term dV_t^2 can be reduced to[4]:

$$dV_t^2 = \sigma^2 V_t^2 dt \qquad (3)$$

By substituting 1 and 3 into equation 2 and incorporating the portfolio replication, one can drive for the Black & Scholes partial differential equation whose solution will depend on the specified boundary condition[5]:

$$\frac{\partial F}{\partial V_t} RV_t + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 F}{\partial V^2_t} - RF \quad = 0 \qquad (4)$$

W_T is normally distributed with zero mean and variance *T*, default probabilities *p(T)* are given by

$$p(t) = P[V_t < K] = P[\sigma W_t < \log L - mT] = N\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right)$$

Where $L = \frac{K}{V_0}$ is the initial leverage ratio and *N(.)* is the standard normal distribution function such that $N(x) = \int_{-\infty}^{x} \exp\left(-\frac{1}{2}z^2\right) dz$

If at time *T* the firm's asset value exceeds the promised payment *K*, the lenders are paid the promised amount and the shareholder receive the residuals asset value. In the even the asset value is less than the promised payment the firm defaults and the ownership of the firm will be transferred to the bond holders. Equity is worthless because of limited liability[6]. The value of the bond issue B_T^T at time *T* is given by

$$B_T^T = \min(K, V_T) = K - \max(0, K - V_T)$$

The above payoff is equivalent to of a portfolio composed of a default-free loan with face value *K* maturing at time *T* and a short European put position on the assets of the firm with strike *K* and maturity *T*. The value of the equity is equivalent to the payoff of a European call option on the assets of the firm with strike *K* and maturity *T*.

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[3] This means that the drift and the diffusion parameters are not too irregular. Square integrability would satisfy this condition.
[4] See Bjork Tomas "Arbitrage theory in continuous time" (1998). Oxford University Press
[5] See Wilmott, Howison and Dewynne "The mathematics of financial derivatives" (1995). Cambridge University Press
[6] See Table 2.1

Table 2.1 Payoffs at maturity

	Assets	Bonds	Equity
No Default	$V_T \geq K$	K	$V_T - K$
Default	$V_T \leq K$	V_T	0

$$E_T = \max(0, V_T - K),$$

Pricing equity and credit risky debt reduces to pricing European options. Under the classical Black-Scholes setting with constant risk free rate, volatility and solving equation *(4)* by imposing suitable initial and boundary conditions, the equity value is given by the Black-Scholes vanilla call option formula:

$$E_0 = V_0 N(d_1) - \exp(-rT)KN(d_2) \tag{5}$$

Where

$$d_1 = \frac{\ln\left(\frac{K}{V_0}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

The value of the corresponding the defaulable bond is given by

$$B_0^T = K\exp(-rT) - P(\sigma, K, T, V, r) \tag{6}$$

Where P is the Black-Scholes put option formula. The value of the put option is equal to the present value of the default loss suffered by bond investors. This is the discount for the default risk relative to the risk-free bond, which is valued at $K\exp(-rT)$. This yields

$$B_0^T = V_0 - V_0 N(d_1) + \exp(-rT)KN(d_2) \tag{7}$$

Using equation *(5)* and *(7)*, the market value of the firm is given by:

$$V_0 = V_0 N(d_1) - \exp(-rT)KN(d_2) + V_0 - V_0 N(d_1) + \exp(-rT)KN(d_2) \tag{8}$$

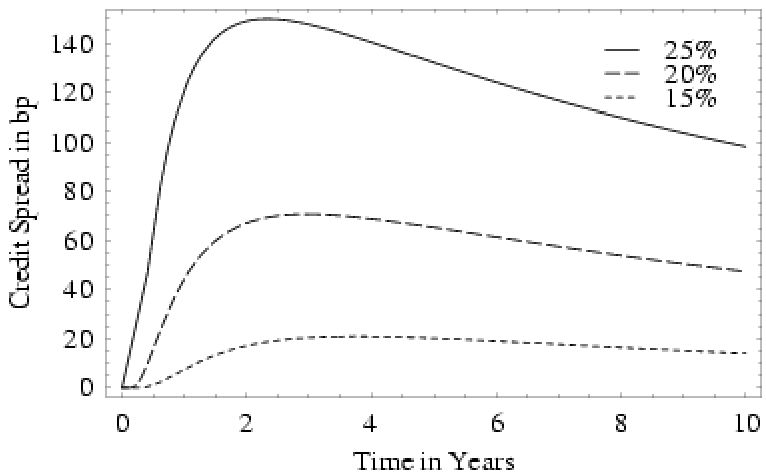
Both of the equity value and debt value will depend on the firm's leverage ratio, equation (8) shows however that their sum does not depend on the firms leverage ratio. This result asserts that the market value of the firm is independent of its leverage, see Rubinstein (2003). The risk neutral default probability can be expressed:

$$p(t) = \frac{\ln\left(\frac{K}{V_0}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} - \sigma\sqrt{T} \tag{9}$$

This depends only on the leverage, L, the asset volatility σ, and maturity time of the debt, T.

The credit spread is the difference between the yield on the defaultable bond and the yield of an equivalent default-free zero coupon bond. This is the access return demanded by bond investors to bear the potential default losses.

Figure 2.2 ***Credit Spread Term Structure***



Term structure of credit spreads with different levels of volatilities
Source: Bloomberg L.P

Since the yield $Y(t,T)$ on a bond with price $b(t,T)$ satisfies;

$$b(t,T) = \exp(-y(t,T)(T-t)$$

The credit spread $S(t,T)$ at time t for a maturity of T is given as:

$$S(t,T) = -\frac{1}{T-t}\log(\left(\frac{B_t^T}{\beta_t^T}\right), \qquad T > t,$$

Where β_t^T is the price of default-free bond maturing at time T. The term structure of credit spreads is the schedule of $S(t,T)$ against different maturities, holding t fixed. In the Black-Scholes framework, we have $\beta_t^T = K\exp(-r(T-t))$ and we obtain the credit spread at time zero

$$S(0,T) = -\frac{1}{T}\log(\left(N(d_2)+\frac{1}{L}e^{rT}N(-d_1)\right), \qquad T > 0,$$

which is a function of maturity *T*, asset volatility σ (the firms business risk), the initial leverage ratio *L*, and the risk free rate *r*. Figure 2.2 plots the credit spread between defaultable and non defaultable bonds with various maturities and different volatility levels. This information can be used as a proxy to derive the prices of credit derivatives contracts such as credit default swaps.

According to Kim, Ramaswamy, and Sundersan (1993) and Jones, Mason, and Rosenfeld (1984), Merton's model does not generate the level of yield spreads which can be observed in the market. They showed that Merton's model is unable to generate yield spreads in excess of 120 basis points, whereas over a period between 1926-1986, the yield spread of AAA-rated corporate ranged from 15 to 215 basis points.

Using a set of structural models, Ericsson, J, Reneby, J and Wang, H., (2005) investigated bond yield spreads and the price of default protection for a sample of US corporations. Theory predicts that if credit risk alone explains these two quantities, their magnitudes should be similar. Their findings are consistent with previous results that bond yield spreads are underestimated. However, their result showed that credit default swap prices (premium) were much lower than bond spreads. Furthermore, their results highlighted the strong relationship between bond residuals and non-default proxies, in particular illiquidity. CDS residuals exhibit no such relations. This suggests that the bond spread underestimation by structural models may not stem from their inability to properly account for default risk, but rather from the importance of the omitted risk factors.
Pricing section of this paper will provide further empirical analysis on credit default swap premium data from the market and default risk pricing from structural models.

2.1.2 Advantages, disadvantages and model extensions

The main advantage of Merton's model is that it allows to directly apply the theory of pricing European options developed by Black and Scholes (1973). However, the model requires to make the necessary assumptions to adapt the dynamics of the firm's asset value process, interest rates and capital structure to the requirements of the Black-Scholes model.
Despite its simplicity and intuitive appeal, Merton's model has many limitations. First, in the model the firm defaults only at the maturity of the debt, a scenario that may not be very realistic in real life. Second, another problem of Merton's model is the restriction of default time to the maturity of the debt, ruling out the possibility of an early default, no matter what happens with the firm's value before the maturity of the debt. If the firm's value falls down to minimal levels before the maturity of the debt but it is able to recover and meet the debt's payment at maturity, the default would be avoided in Merton's approach. Third, Another problem with the Merton model is that the value of the firm, an input to the valuation formula, is very difficult to determine. Unlike the stock price in the Black-Scholes-Merton formula for valuing equity options, the current market value of a firm is not easily observable.

Another limitation of the model is that the usual capital structure of a firm is much more complicated than a simple zero-coupon bond. Geske (1977, 1979) considers the debt structure of the firm as a coupon bond, in which each coupon payment is viewed as a compound option and a possible cause of default. At each coupon payment, the shareholders have the option either to make the payment to bondholders, obtaining the right to control the firm until the next coupon, or to not make the payment, in which case the firm defaults. Geske also extends the model to consider characteristics such as sinking funds, safety covenants, debt subordination and payout restrictions.

The assumption of a constant and flat term structure of interest rates is other major criticism the model has received. Jones et al. (1984) suggest that "there exists evidence that introducing stochastic interest rates, as well as taxes, would improve the model's performance." Stochastic interest rates allow to introduce correlation between the firm's asset value and the short rate, and have been considered, among others, by Ronn and Verma (1986), Kim, Ramaswamy and Sundaresan (1993), Nielsen etal. (1993), Longstaff and Schwartz (1995), Briys and de Varenne (1997) and Saá-Requejo and Santa Clara (1997).

Another characteristic of Merton's model, which will also be present in some of the First Passage Models (FPM), is the predictability of default. Since the firm's asset value is modelled as a geometric Brownian motion and default can only happen at the maturity of the debt, it can be predicted with increasing precision as the maturity of the debt comes near. As a result, in this approach default does not come as a surprise, which makes the models generate very low short-term credit spreads[7].

2.2 First passage model.

First passage model were introduced by Black & Cox (1976) extending the Merton model to the case when the firm may default at any time, not only at the maturity date of the debt. They also assume that the firm's shareholders receive a continuous dividend payment, proportional to the current value of the firm. Consequently as in section 2.1, the SDE which governs the dynamics of the firm's value takes the following form, under the risk neutral probability measure P[8],

$$dV_t = V_t\left((r-k)dt + \sigma dW_t\right) \qquad (10)$$

Where $k \geq 0$ represents the payout ratio (continuous dividend payment), $\sigma > 0$ and r represent constant volatility and constant short term interest rate respectively.

Safety covenant in the firm's debt prospectus give the bondholders the right to force the firm to bankruptcy if the firm is doing poorly according to a set standard. The standard

[7] See Jones et al. (1984) and Franks and Torous (1989).

[8] The drift term in equation 1 is now adjusted to dividend payout. In this section, some of the notations introduced in previous sections are changed but their interpretation still remain the same.

for a poor performance is set in Black and Cox in terms of a time-dependent deterministic barrier:

$$\bar{v}(t) = Ke^{-\gamma(t-T)}, \quad t \in [0,T), \tag{11}$$

With some constant *K*. when the value of the firm crosses this lower threshold the bondholders takeover the firm. Otherwise default takes place at maturity of debt depending on whether or not $V_T \geq L$, where *L* represents (see footnote 7) the face value of the firm's debt.

$$v_t = \begin{cases} \bar{v}(t) & for \quad t < T, \\ L & for \quad t = T. \end{cases}$$

The default time τ is the first moment in the interval *[0, T]* when the firm's value V_t falls below the time varying level v_t; otherwise the default event does not occur at all. The default time τ can be defined as

$$\tau = \inf\{t \in [0,T]:\ V_t < v_t\},$$

Formally, we deal with the defaultable contingent claim $(X, Z, \bar{X}, \tau)$ which settles at time *T*; where

$$X = L,\ Z_T = \beta_2 V_t,\ \bar{X} = \beta_1 V_T.$$

The random variable *X* represents the firms liabilities to be redeemed at time *T* (promised claim). If default does not occur prior to or at time *T*, the promised claim *X* is paid in full at time *T*. otherwise either:

(i) *default occurs at time t < T, and the holder of the defaultable claim receives the recovery payoff* Z_t *at time t, or:*

(ii) *default occurs at the debt maturity T, and the recovery payoff* $\bar{X}$ *is received by the claimholder at time T.*

The recovery process Z is assumed to be proportional to the firm's value process: $Z_t = \beta_2 V_t$ for some constant β_2. Similarly, the recovery payoff at maturity equals $\bar{X} = \beta_1 V_T$ for some constant β_1. The coefficient β_1 and β_2 are constant and represent the bankruptcy costs. The default time $\tau = \bar{\tau} \wedge \hat{\tau}$ where $\bar{\tau}$ is the passage time of the firm's value process *V* to the deterministic barrier $\bar{v}$:

$$\tau = \inf\{t \in [0,T]:\ V_t \leq \bar{v}(t)\} = \inf\{t \in [0,T]:\ V_t < \bar{v}(t)\}$$

$\hat{\tau}$ is the Merton's default time:

$$\hat{\tau} = T1_{\{V_T < L\}} + \infty 1_{\{V_T \geq L\}}.$$

Assuming that for any $t \in [0,T)$ we have $\bar{v}(t) \leq LB(t,T)$, where *B(t, T)* is the price of defaultable zero coupon bond at time t with a maturity of *T*.

$$Ke^{-\gamma(t-T)} \leq Le^{-r(T-t)} = LB(t, T).$$

Thus the payoff to the bondholder at the default time τ never exceeds the value of debt discounted at the risk-free rate. At time *t < T,* the value of a defaultable zero-coupon bond with face value of *L* and maturity *T,* denoted by *D(t, T),* admits the following probabilistic representation, $\{\tau > t\}$ *and* $= \{\bar{\tau} > t\}$,

$$D(t,T) = E_P\left(Le^{-r(T-t)} 1_{\{\bar{\tau} \geq T,\, V_T \geq L\}} \mid F_t\right) \qquad \text{no default}$$

$$+ E_P\left(\beta_1 V_T e^{-r(T-t)} 1_{\{\bar{\tau} \geq T,\, V_T < L\}} \mid F_t\right) \qquad \text{default at } T$$

$$+ E_P\left(\beta_2 K V_T e^{-\gamma(T-\bar{\tau})} e^{-r(\bar{\tau}-t)} 1_{\{t < \bar{\tau} < T\}} \mid F_t\right) \qquad \text{default at } t < \tau < T \qquad (12)$$

After evaluating the above expectations, Bingham and Kiesel show a closed-form solution for the value of a defaultable zero-coupon bond as a down-and-out barrier option[9]. From this model, one can then infer the default probability from time *t* to *T*:

$$P[\tau \leq T \mid \tau > T] = N(h_1) + \exp\left(2\left(r - \frac{\sigma_V^2}{2}\right)\ln\left(\frac{K}{V_t}\right)\frac{1}{\sigma_V^2}\right)N(h_2), \qquad (14)$$

Where

$$h_1 = \frac{\ln\left(\dfrac{K}{e^{-r(T-t)V_t}}\right) + \dfrac{\sigma_V^2}{2}(T-t)}{\sigma_V \sqrt{T-t}}, \qquad (15)$$

$$h_2 = h_1 - \sigma_V \sqrt{T-t,} \qquad (16)$$

The first passage model have been extended to account for stochastic interest rates, bankruptcy costs, taxed, debt subordination, strategic default, time dependent and stochastic default barrier, jumps in the asset value process, etc. These extensions may take into account several important market related factors but these improvements adds significant complexity to the model. See Bielecki and Rutrkowski (2003) for more in-depth analysis. Further discussion on calibration of the first passage model and application to credit default swap market price quotes, see; Damiano Brigo and Marco

[9] Interested readers should refer to Bingham N.H., and Kiesel, Rudiger "Risk-Neutral Valuation: Pricing and hedging of financial derivatives" Spring Finance, 2nd Edition.

Tarenghi (2005) “credit default swap calibration and counter party risk valuation with a scenario based first passage model” Working paper.

2.2.1 Other Structural Models

2.2.2 The Kim, Ramaswamy and Sundaresan 1993 Model

Kim, Ramaswamy and Sundaresan (1993) used simpler default boundary but more realistic stochastic interest rate process the Black-Cox 1976 model. Default is triggered if the asset value drops below an exogenous constant *w*. The interest rate process follows the risk-neutral Cox-Ingersoll-Ross Model:

$$dr = a(b-r)dt + \sigma_1\sqrt{r}\,dW_1 \qquad (17)$$

Where *r* is the interest rate, *a* is mean reversion factor and *b* is long term average of *r*, sigma is the volatility of *r* and *W* is Weiner process. This model has the convenient property that the interest cannot become negative.

2.2.3 The Longstaff-Schwartz 1995 Model

Longstaff and Schwartz suggest a first-passage model with exogenous and constant default boundary *K* and an exogenous and constant recovery rate *w*. For the interest rate process, Longstaff and Schwartz use the well-known Vasicek model:

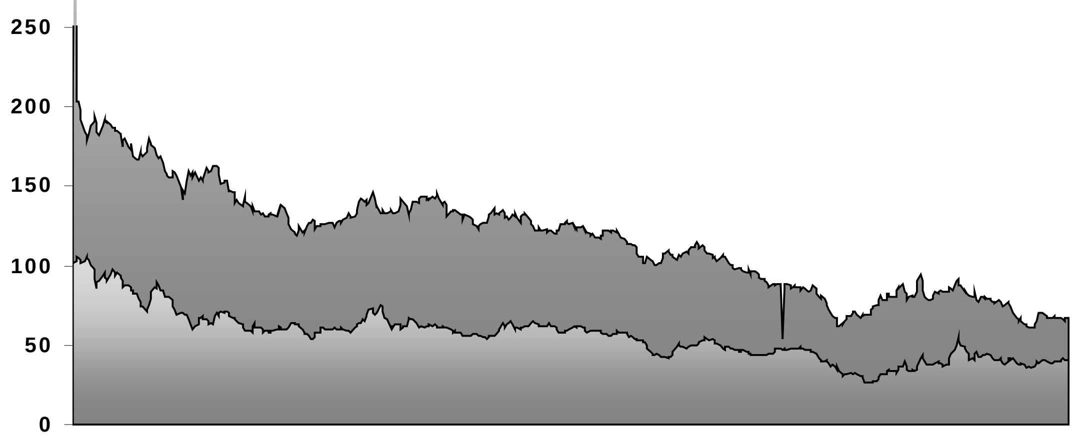
$$dr = a(b-r)dt + \sigma_1\eta dW_1 \qquad (18)$$

Where η is an exogenous constant and all the other parameters are the same as in the Cox-Ingersoll-Ross Model. One of the key findings of Longstaff and Schwartz model is that the long term average of the interest rate can be negative *(b < 0)*. This implies that the credit spreads decrease when the risk-free treasury rate increases. This seems counterintuitive but can be explained by the fact that a higher interest rate implies higher growth rate of the asset value *V* and as a consequence the probability of default is lower, and with it the credit spread.

2.3. Empirical tests from structural models

Hull, Nelken and White (2203) tested whether five year credit spreads implied from their implementation of Merton's model and the traditional implementation are consistent with market quoted credit default swaps (CDS) premiums. There are number of reasons for the difference between the credit spreads implied from Merton's model and the market observed credit default swaps spreads. According to Hull, Nelken and White (2203), Merton's model is not a perfect representation of market practise because firm's do not usually issue only zero-coupon debt. Credit default swap spreads are also likely to be slightly different from the bond yield spreads for some of the reasons listed in section 5.4.

Figure 2.3 ***Historical 5 year CDS spread vs Credit spread for France Tel. (2003-2005)***



Source: Bloomberg LP. July 2005

The credit spread backed out from Merton's model is the spread between the yield on zero-coupon bonds while the credit default swap spreads are based on spread between the yields on par yield bonds. The authors also suggest that there may be other factors other than those suggested by Merton's model may affect the CDS spreads. This is similar to the conclusion drawn from the implementation of the reduced-form models. These factors will be discussed briefly (see section 5.4).

3.1 Reduced-Form Credit Risk Models.

3.1.1 *Intensity Based Model.*

Reduced form models do not model the evolution of the firm's value. Instead a specified jump process models default exogenously. There are two classes of reduced form models. They are intensity-based models that are concerned with modeling the time of the default event and credit migration models that are concerned with modeling the migration between credit ratings. Credit derivative models use intensity-based models, as it is the modelling of the default event that is important in pricing credit derivative contracts. A common problem with structural models is that default may occur before the boundary conditions have been met and at other times it may not occur even when the boundary conditions have been met. Intensity-based models therefore try to model the likelihood of default rather than trying to specify the actual time of default. According to Bielecki and Rutkowski (2002) intensity-based models were first formalized by Jarrow and Turnbull (1995) and Madan and Unal (1998). In intensity based models the time of default is modeled as the time of the first jump of a Poisson process with (possibly random) intensity. The time of default is denoted by τ and can be regarded as a stopping time. At the stopping time τ the default indicator function *I(t)* jumps from zero to one and is denoted by

$$I(t) \quad = \quad 1_{\{\tau \le t\}} = \begin{cases} 1 & if \quad \tau \le t \\ 0 & if \quad \tau > t. \end{cases}$$

In general, there could exist more than one stropping time τ before time *t*. Assuming that $\tau_i < \tau_{i+1}$ the collection of stopping times is given by the point process

$$\{\tau_i, i \in \mathrm{N}\} = \{\tau_1, \tau_2, ...\}.$$

The counting process *N(t)* counts the number of stopping times of the point process that are before time *t*, and is given by

$$N(t) = \sum_{i=1}^{\infty} 1_{\{\tau_i \le t\}}.$$

In reality there can only be one default event and so the time to default τ is the time of the first jump of *N*,

$$\tau = \inf\{t \in \Re_+ \mid N(t) > 0\},$$

The probability of *N(t)* jumping in an infinitesimally small time interval is called the default intensity function $\lambda(t)$ or the *hazard rate*. In order to determine the probability of a default occurring in the time interval *[0, T]*, it is useful to recap a few properties of the Poisson process. Firstly, by the definition of a Poisson process the probability of *n* jumps occurring in the time interval *[0, T]* is given by;

$$P[N(t) - N(0) = n] = \frac{1}{n!}\left(\int_0^T \lambda(s)ds\right)^n \exp\left(-\int_0^T \lambda(s)ds\right). \qquad (19)$$

Secondly, this implies that the probability of no jumps occurring in the interval *[0, T]* is given by

$$P[N(t) - N(0) = 0] = \exp\left(-\int_0^T \lambda(s)ds\right) = P(0,t) \qquad (20)$$

and, therefore, *P(0, t)* is the probability that no default events occur in the interval *[0, T]* and is called the survival probability. The probability that a default event occurs in the interval is equal to one minus the probability of surviving. Hence the probability of default P^{def} in the interval *[0, t]* is given by

$$P^{def}(0,T) = 1 - P(0,t) = 1 - \exp\left(-\int_0^T \lambda(s)ds\right) \qquad (21)$$

The intensity approach to modeling credit risk was studied by, among others, Jarrow & Turnbull (1995), Jarrow et al. (1997), Duffie et al. (1996), Duffie (1998a), Lando (1998a), Duffie and Singleton (1999), Elliot et al. (2000), Schonbucher (2000a, 2000b).

3.1.2 Jarrow and Turnbull (1995) – discrete approach.

Jarrow & Turnbull use the risk-free rate as numeraire, they build a discrete lattice for the default-free term structure as well as for defaultable one and show that, under this numeraire, they can obtain unique risk-neutral or martingale probabilities such that the value of defaultable bond can be expressed as a discount expectation under the risk-neutral measure. In this section we drive a simplified mathematical formulation of the default probability with no recovery rate using one period discrete time.

Define:

$y(T)$:	Yield on a T-year corporate zero-coupon bond
$y^*(T)$:	Yield on a T-year risk-free zero-coupon bond
$Q(T)$:	Probability that a corporation will default between time zero and T or $Q_T = Q(\tau < T)$ risk neutral probability of default before time T $Q(\tau > T) = (1 - Q_T)$ survival probability
τ :	Default time
$P_{t,T}$	Default free zero-coupon bond
$P^d{}_{t,T}$	Defaultable zero-coupon bond

The present value of a T-year risk-free zero-coupon with a redemption value of 100 is $100e^{-y^*(T)T}$ while the present value of a corporate zero-coupon bond with similar maturity is $100e^{-y(T)T}$. The expected loss from default is therefore

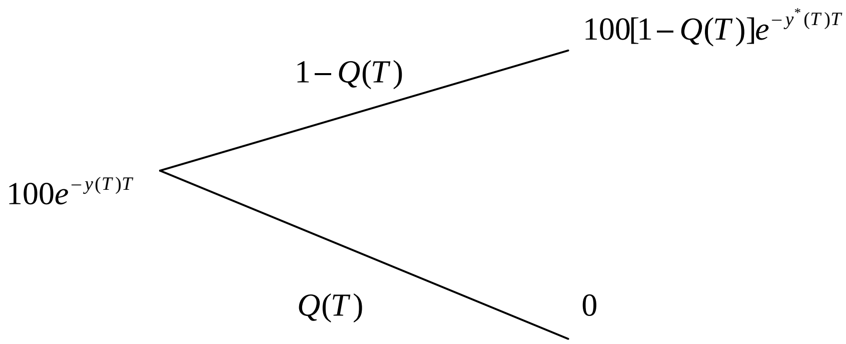
$$100e^{-y^*(T)T} - 100e^{-y(T)T} = 100[e^{-y^*(T)T} - e^{-y(T)T}]$$

Using the indicator function, the expected payoff of the defaultable bond can be written as:

$$II_{\tau>T} = \begin{cases} 100 & if \quad \tau > T, \\ 0 & if \quad \tau < T. \end{cases}$$

If we assume that there is no recovery in the event of default, the calculation of $Q(T)$

is relatively easy. At maturity of the corporate bond, it will either be worth zero if default takes place or 100 (par value of the bond) with probabilities of $Q(T)$ or $1-Q(T)$ respectively. Lets build a two state binomial tree from period 0 and T, where the node at *t = 0* is the present value of the corporate bond, and nodes at period *t = T* are the expected payoff of the bond if default takes place or not. From the tree below, we can derive the risk-neutral default probabilities between period zero and *T*.



The value of the bond can be calculated as

$$100e^{-y(T)T} = 100[1-Q(T)]e^{-Y*(T)T} + [100Q(T)]0$$

$$100e^{-y(T)T} = 100[1-Q(T)]e^{-Y*(T)T}$$

$$100e^{-y(T)T} = 100[e^{-y*(T)T} - Q(T)e^{-Y*(T)T}]$$

$$\frac{100e^{-y(T)T}}{100} = \frac{100[e^{-y*(T)T} - Q(T)e^{-y*(T)T}]}{100}$$

$$e^{-y(T)T} = e^{-y*(T)T} - Q(T)e^{-Y*(T)T}$$

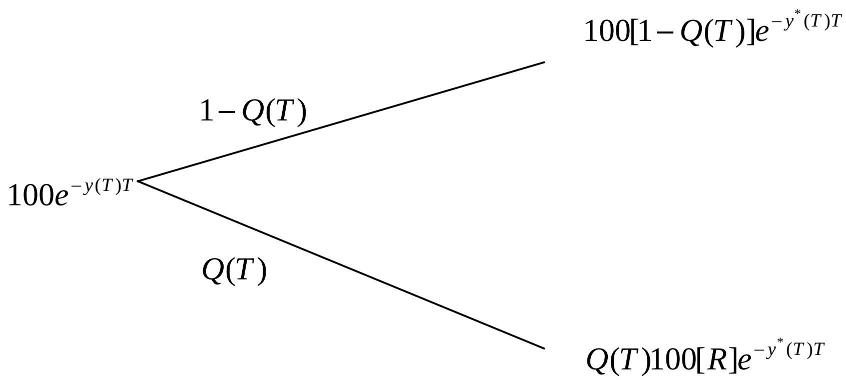
$$Q(T)e^{-Y*(T)} = e^{-y*(T)T} - e^{-y(T)T}$$

$$\frac{Q(T)e^{-y*(T)T}}{e^{-y*(T)T}} = \frac{e^{-y*(T)T} - e^{-y(T)T}]}{e^{-y*(T)T}}$$

$$Q(T) = \frac{e^{-y*(T)T} - e^{-y(T)T}]}{e^{-y*(T)T}} \qquad \text{or} \qquad Q(T) = 1 - e^{-[y(T)T - Y*(Y)T\}} \qquad (22)$$

Probability of Default assuming recovery rate R:

If we use the same notation as before and suppose the expected recovery rate is *R*, then if default takes place, the bondholder receives a proportion of *R* of the bond's no-default value. If there is no default, the bondholder receives the face value of the bond (100). Lets set up the same tree again but incorporate recovery rate of *R* this time.



We can derive the value of the bond as

$$100e^{-y(T)T} = 100[1-Q(T)]e^{-y^*(T)T} + \ Q(T)100[R]e^{-y^*(T)T}$$

Now, let $\quad {}^{-y(T)T} \ = \ A$

$\qquad\qquad {}^{-y^*(T)T} = \ B$

Then,

$$100e^{-A} = 100[1-Q(T)]e^{-B} \ + \ Q(T)100[R]e^{-B}$$

$$100e^{-A} = 100e^{-B} - Q(T)100e^{-B} + Q(T)100\,\mathrm{Re}^{-B}$$

$$100e^{-A} = 100e^{-B} - Q(T)100e^{-B}[1-R]$$

$$Q(T)100e^{-B}[1-R] = 100e^{-B} - 100e^{-A}$$

$$Q(T) = \frac{100e^{-B} - 100e^{-A}}{100[1-R]e^{-B}} \ = \ Q(T) = \frac{e^{-B} - e^{-A}}{[1-R]e^{-B}}$$

now, substituting A and B for the original values, we have the solution

$$Q(T) = \frac{e^{-y^*(T)T} - e^{-y(T)T}]}{[1-R]e^{-y^*(T)T}} \qquad or$$

$$Q(T) = \frac{1-e^{-[y^*(T) \ - -y(T)]T}}{e^{-y^*(T)T}} \tag{23}$$

Table 1.2

	PRICE	COUPON	DISC.FACT YEAR 1	DISC.FACT YEAR 1	DISC.FACT YEAR 1
Defaultable Par Bond 1	100	8	0.9259		
Defaultable Par Bond 2	100	10		0.8249	
Defaultable Par Bond 3	100	12			0.70527
Benchmark Swap Curve	Price	Coupon	3	4	5
	PRICE	COUPON	DISC.FACT YEAR 1	DISC.FACT YEAR 2	DISC.FACT YEAR 3
Default Free Bond 1	100	3	0.9709		
Default Free Bond 2	100	4		0.9335	
Default Free Bond 3	100	5			0.88908
			YEAR 1	YEAR 2	YEAR 3
Discount Spreads			0.0449	0.1086	0.18381
Risk Neutral Default Probability			0.09259	0.15032	0.219393

From table 1.2, we drive the risk-neutral default probability for defaultable bond from 1 year to 3 years using par defaultable and risk-free bonds with coupon payments. We use the following notations to explain the above result[10]:

Define:

$SP(T)$	Discount factor spread for period *T*
$Q(T)$:	Default probability for period *T*
R :	Recovery rate
$DF_B(T)$	Benchmark discount factor for period *T*

The default probabilities for period 1 to 3 are calculated as

$$Q(1) = (SP(1)/(1-R)*DF_B(1))$$
$$Q(2) = (SP(2)-SP(1))/(1-Q(1))(1-R)*DF_B(2))$$
$$Q(3) = (SP(3)-SP(2)/(1-Q(1))(1-Q(2))(1-Q(3))(1-R)*DF_B(3))$$

Table 1.3

MATURITY	RISK-FREE ZERO -RATE	CORPORATE BOND ZERO- RATE	EXPECTED DEFAULT LOSS - (% OF NO DEFAULT VALUE)	CUMULATIVE DEFAULT PROBABLTY	Default Probability
1	0.05	0.0525	0.2497%	0.2497%	0.2497%
2	0.05	0.055	0.9950%	0.9950%	0.7453%
3	0.05	0.057	2.0781%	2.0781%	1.0831%
4	0.05	0.0585	3.3428%	3.3428%	1.2647%
5	0.05	0.0595	4.6390%	4.6390%	1.2961%

Default probability can be quantified in terms of default probability density or in terms of hazard rate. The (risk neutral) hazard rate $\gamma = \gamma(t)$ is defined by

[10] The data in table 1.2 and 1.3 are numerical implementation from excel spreadsheet

$$\gamma(t)dt = Q(t < \tau \le t + dt \mid \tau > t),$$

Which is the probability of default between time *t* and *t+dt* conditional on no earlier default. $\gamma(t)dt$ is the likelihood of default between time *t* and time *t+dt* conditional on no default between time zero and time *t.* The default probability density $q(t)dt$ is the unconditional default probability between times *t* and time *t+dt* conditional as seen at time zero and the relationship between $\gamma(t)$ and *q(t)* is:

$$q(t) = h(t)e^{-\int_0^t h(\tau)d\tau}$$

Table 1.3 lists the unconditional probabilities of default as seen at time zero. The default probability for year 5 is computed as 1.2921%. The hazard rate is the default probability in year 5 given that no default has taken place up to year 4. The probability of no default prior to year 4 is 1 – 0.0033428 = 0.966572%. The hazard rate for year 4 is therefore 0.012962/0.966572 = 1.3410%

3.2. Arbitrary deterministic recovery, deterministic interest rates.

If the recovery rate at the random default time τ is $Z(\tau)$, where the function $Z = Z(t)$ is some deterministic function of time, and also that the risk-free interest rate *r(t)* depends deterministically on time. The price of defaultable bond via risk-neutral pricing is then given by,

$$P_{0,T}^{d} = E_Q\left(e^{-\int_0^T r(u)du} \; 1_{\tau > T} + e^{-\int_0^\tau r(u)du} \; Z_\tau \; 1_{\tau \le T} \right). \qquad (24)$$

To compute this, one needs the probability distribution of the default time τ. Instead of working directly with $Q(\tau > T)$, we use a closely related quantity, the *hazard rate* which is specified above.

$$\gamma(t)dt = Q(t < \tau \le t + dt \mid \tau > t),$$

The relationship between the hazard rate and the cumulative distribution function (cdf) of τ is described:

Lemma 3.1. For all $t \ge 0$, we have that

$$Q(\tau > t) = e^{-\int_0^t \gamma(u)du},$$

Consequently, the cdf and the pdf of τ with respect to the probability *Q* are,

$$F_\tau(t) = F_\tau^Q(t) = Q(\tau \le t) = 1 - e^{-\int_0^t \gamma(u)du}$$

And

$$f_\tau(t) = f_\tau^Q(t) = \gamma(t)e^{-\int_0^t \gamma(u)du}$$

Which is the derivative of F_τ [11]

Now revisiting the zero-coupon bond price of a defaultable bond with zero recovery but with deterministic and time dependent interest rate,

$$payoff = \begin{cases} 1 \quad if \quad \tau > T, \\ 0 \quad if \quad \tau < T. \end{cases}$$

Then;

$$P_{0,T}^d = e^{-\int_0^T r(u)du} Q(\tau > T) + 0.Q_t =$$

and

$$P_{0,T}^d = e^{-\int_0^T r(u)du + \gamma(u)du} \qquad (25)$$

That is, instead of discounting with *r(t),* we discount with *r(t)* and $\gamma(t)$. where $\gamma(t)$ is an instantaneous spread.

If we now assume a deterministic recovery $Z(\tau)$ at time τ, that is, the amount of $Z(t)$ recovered should default time happen at t $(t = \tau)$ is known beforehand; the quantity $Z(\tau)$ is random variable since τ is a random variable. As before:

$$P_{0,T}^d = E_Q\left(e^{-\int_0^T r(u)du} \; 1_{\tau > T} + e^{-\int_0^\tau r(u)du} \; Z_\tau \; 1_{\tau \le T} \right). \qquad (26)$$

The discount factor $e^{-\int_0^T r(u)du}$ is pulled out of the expectation sign since *r(t)* is assumed to be deterministic. However the second discounting factor is stochastic because of the random time τ and cannot be pulled out of the expectation sign. As specified above the first discount factor in equation *(26)* is adjusted; $e^{-\int_0^T r(u)du + \gamma(u)du}$ and we evaluate the remaining expression inside the expectation sing with the random component (second term in equation *(27):*

$$E_Q\left(e^{-\int_0^\tau r(u)du} \; Z_\tau \; 1_{\tau \le T} \right).$$

We had an expression for the probability of default between *(t, t+dt)*:

[11] The proof of this result, see pricing notes for Financial Engineering lecture notes by Professor Brummelhuis, R. Birkbeck University.

$$f_t^Q = Q(t < \tau \le t + dt) = \gamma(t)Q(\tau > t)dt = \gamma(t)e^{-\int_0^t \gamma(u)du}$$

$$E_Q\left(e^{-\int_0^\tau r(u)du} \quad Z_\tau \quad f_t^Q \right),$$

$$E_Q\left(e^{-\int_0^\tau r(u)du} \quad Z_\tau \quad \gamma(t)e^{-\int_0^t \gamma(u)du} \right).$$

The last equation cane be written as;

$$E_Q\left(Z(s) \;\; \gamma(s)\; e^{-\int_0^s r(u)du+\gamma(u)du} \;\; ds \right).$$

Then;

$$P_{0,T}^d = \left(e^{-\int_0^T r(u)du+\gamma(u)du} + \int_0^T Z(s) \;\; \gamma(s)\; e^{-\int_0^s r(u)du+\gamma(u)du} \;\; ds \right). \qquad (27)$$

And the price at time *t;*

$$P_{t,T}^d = 1_{\tau>t}\left(e^{-\int_0^T r(u)du+\gamma(u)du} + \int_0^T Z(s) \;\; \gamma(s)\; e^{-\int_0^s r(u)du+\gamma(u)du} \;\; ds \right). \qquad (28)$$

3.2.1 Stochastic recovery, interest rate and hazard rate

In stochastic interest rate environment, and with stochastic recovery $Z(\tau)$ and hazard rate $\gamma(t).$, one has:

$$P_{t,T}^d = 1_{\tau>t}\left\{ E_Q\left(e^{-\int_0^T r(u)du+\gamma(u)du} \mid f_t \right) + E_Q\left(\int_t^T Z(s) \;\; \gamma(s)\; e^{-\int_0^s r(u)du+\gamma(u)du} \;\; ds \mid f_t \right) \right\} \qquad (29)$$

For more rigorous proof, see Bielecki and Rutkowski (2002) "Credit Risk" Springer Verlag.

In case of zero recovery, the last term of the above equation drops out and we are left with;

$$P_{t,T}^d = 1_{\tau>t}\left\{ E_Q\left(e^{-\int_0^T r(u)du+\gamma(u)du} \mid f_t \right) \right\} \qquad (30)$$

3.2.2 Recovery Protocols

In the real world payoffs of defaulted securities are usually greater then zero. The recovery rates (given default), denoted by $Z(\tau)$, is defined as the extent to which the value of an obligation can be recovered once the obligor has defaulted, i.e. the recovery rate is a measure for the expected fractional recovery in case of default and such that it takes any value in the interval *[0, T]*. The loss rate (given default), is defined as 1 minus recovery rate.

Fractional Recovery of Par:

It is assumed that there is a compensation in terms of cash (invested in risk-free money market account) and the recovery rate is expressed as a fraction of par. The model has been applied, e.g., by Duffie (1998b).

If V represents the claims constant par value and δ is the claims recovery rate, then;
$Z_t = V\delta$ and

$$P^d_{t,T} = 1_{\tau>t}\left\{E_Q\left(e^{-\int_t^T r(u)du+\gamma(u)du} V \mid f_t\right) + E_Q\left(\delta V\int_t^T Z(s)\ \gamma(s)\ e^{-\int_t^s r(u)du+\gamma(u)du}\ ds \mid f_t\right)\right\} \quad (31)$$

Fractional Recovery of treasury:

It is assumed that there is a compensation in terms of (the value of) non-defaultable bonds, i.e. the value of equivalent treasury bond. Several authors have proposed this model, e.g., Jarrow & Turnbull (1995), Madan & Unal (1998).

In the even of default, recovery is then given by: $Z_\tau = \delta\, e^{-\int_\tau^T r(u)du}$ and the price of a defaultable bond using this assumption;

$$P^d_{t,T} = 1_{\tau>t}\left\{E_Q\left(e^{-\int_t^T r(u)du+\gamma(u)du} V \mid f_t\right) + E_Q\left(\delta\int_t^T\ \gamma(s)\ e^{-\int_t^T \gamma(u)du} \mid f_t\right)\right\} \quad (32)$$

Fractional Recovery of market value:

It is assumed that there is a compensation in terms of equivalent defaultable bonds, which have not defaulted yet, i.e. the recovery rate is expressed as a fraction of the market value of the defaulted bond just prior to default. This model was mainly developed by Duffie & Singleton (1997).
Assume Z_τ is a positive fraction of the market value of the bond just prior to default $Z_\tau = \delta\, P^d_{t,\tau}$, then;

$$P_{t,T}^{d} = \left\{ E_Q \left(e^{-\int_t^T r(u)du + s(u)du} \mid f_t \right) \right\} \qquad (33)$$

Where $s(u) = (1-\delta_u)\gamma_u$, and δ_u is the loss rate and γ_u is the intensity of default.

4.1 *Jarrow, Lando and Turnbull (1997) model – discrete approach.*

Jarrow, Lando and Turnbull model is a Markov model for the term structure of credit spreads based on the earlier Jarrow and Turnbull (1995) paper, but linking the default process to a discrete state space Markov chain in credit ratings, i.e. the life of the firm is viewed as a journey through the possible rating states where one of them is an absorbing state. This model provides great flexibility to calculate the parameters to observable data and to use it for many purposes: pricing and hedging of bonds with embedded options, pricing of credit derivatives. The main assumption of this approach is that ratings are an accepted indicator of a firm's creditworthiness. Default is exogenous process that does not require dependence of the underlying asset of the firm. The advantage of such methods against the structural models introduced above is that we can restrict the calibration to the available observables; there is no particular economic requires. Here are a list of all the ingredients for this model:

Forward rates are defined in discrete time as

$$f(t,T) = -\ln\left(\frac{p(t,\, T+1)}{p(t,T)} \right).$$

Under this construction, the instantaneous interest rate *r(t)* is equivalent to *f(t, T)*.
The money market account value is similarly given by

$$B(t) = \exp\left(\sum_{t=0}^{t-1} r(i) \right)$$

Under the assumption of complete arbitrage-free markets, we have the following relationship:

$$B(t) = \overline{E}_t \left(\frac{B(t)}{B(T)} \right).$$

And the price of defaultable bond taking into account the default likelihood is they given by:

$$P_{t,T}^{d} = \overline{E}_t \left(\frac{B(t)}{B(T)} \left(\delta 1_{\tau \le T} + 1_{\tau > T} \right) \right). \qquad (34)$$

if default takes place, the payoff is assumed to be fractional recovery of treasury as specified by Jarrow and Turnbull.[12] Since the default free term structure and the default time are assumed to be independent, then;

$$\begin{aligned} P_{t,T}^{d} &= \overline{E}_t\left(\frac{B(t)}{B(T)}\right)\overline{E}_t\left(\delta 1_{\tau\le T} + 1_{\tau>T}\right) \\ &= p(t,T)(1-\delta)Q_t(\tau > T)), \end{aligned} \qquad (35)$$

Where $Q_t(\tau > T)$ is the probability that the firm will not default before the maturity (survival probability).

The contribution of Jarrow et al. (1997) articulates around the specification of the bankruptcy process as the first hitting time of a time-homogenous Markov chain. This Markov chain is modelled on a finite state space *S* consisting in the credit rating classes *{1, ...,K}*, where the *K-1* class is the lower credit rating class, while class *K* is the absorbing state representing the bankruptcy state. This Markov chain is specified by *KxK* transition matrix:

$$Q = \begin{pmatrix} q_{11} & q_{12} \cdots & q_{1k} \\ q_{21} & q_{22} \cdots & q_{2k} \\ . & & \\ . & & \\ . & & \\ q_{k-1,1} & q_{k-1,2} \cdots & q_{k-1,k} \\ 0 & 0 \quad \ldots & 1 \end{pmatrix}, \qquad (36)$$

Where all transition probabilities are positive and $= q_{ii} \equiv \sum_{\substack{j=1 \\ i\neq j}}^{K} q_{ij}, \quad \forall i$.

Each of the q_{ij} probabilities represent the probability of getting from class *i* to class *j* in one period of time. The last line in equation *(36)* represents the probabilities attached to the absorbing state: the probability of leaving this state is always null and the probability of staying in this state in 1. Once a firm is in default, it will stay in default. Estimates of these transition probabilities can be found in the reports of credit rating agencies such as Moody's or Standard and Poor.

[12] See recovery protocols above to derive the payoff.

Table 1.4 Credit rating transition matrix

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	94.328	5.015	0.551	0.062	0.040	0.004	0.001	0.0002
AA	0.516	92.337	6.410	0.569	0.089	0.062	0.013	0.0048
A	0.084	1.924	92.442	4.919	0.436	0.174	0.014	0.0082
BBB	0.040	0.265	4.378	89.990	4.376	0.747	0.090	0.1148
BB	0.027	0.091	0.607	5.439	84.497	8.015	0.766	0.5581
B	0.004	0.080	0.273	0.458	4.284	85.367	5.192	4.3420
CCC	0.082	0.011	0.350	0.536	1.131	7.358	50.278	40.255
D	0	0	0	0	0	0	0	100

Source: Moody's 2001

From table 1.4, q_{ij} is the historical ("real-world") probability of moving from credit rating class i to class j in one year as stated above. As mentioned by Jarrow et al., nonzero probabilities tend to concentrate on the diagonal for a 1-year transition matrix since a movement of more than one rating class is quite improbable.

Following from equation *(36)*, the transition matrix under the equivalent martingale[13] measure can be written as:

$$Q_{t,t+1} = \begin{pmatrix} q_{11}(t,t+1) & q_{12}(t,t+1) \; \ldots & q_{1k}(t,t+1) \\ q_{21}(t,t+1) & q_{22}(t,t+1) \; \ldots & q_{2k}(t,t+1) \\ . & & \\ . & & \\ . & & \\ q_{k-1,1}(t,t+1) & q_{k-1,2}(t,t+1) \; \ldots & q_{k-1,k}(t,t+1) \\ 0 & 0 \; \ldots & 1 \end{pmatrix}, \qquad (37)$$

Where

$$q_{ij}(t,t+1) \geq 0, \; \forall i, j, \;\; i \neq j \;\; and \;\; q_{ii}(t,t+1) \equiv 1 - \sum_{\substack{j=1 \\ i \neq j}}^{K} q_{ij}(t,t+1).$$

Additionally

$$q_{ij}(t,t+1) > 0, \; iff \; q_{ij} > 0 \;\; for \;\; 0 \leq t \leq \tau - 1.$$

In the Jarrow and Turnbull (1995) model, default probabilities and credit derivative prices were derived on the basis of illiquid bond prices. However, the Jarrow, Lando, and

[13] For detailed introduction on Martingale representation theorem, Bjork, T. "Arbitrage theory in continuous time." (1989) oxford university press.

Turnbull (1997) model replaced bond prices as the main input and apply historical transition probabilities as the basis for their analysis. Today, many investment banks and insurance companies apply the 1997 model and its extensions to price and hedge credit derivatives. Some of the short comings of this model is that asset-liability structure of a company, is not part of the analysis (this may be argued to be the ultimate economic reason of default). Also, interest rate process and bankruptcy process are assumed to be independent. Jarrow, Lando and Turnbull also assume that bonds in the same credit class have the same yield spread. Longstaff and Schwartz (1995) pointed out that this was not necessarily the case. Rating are also done infrequent and may not be recent enough to reflect current counter party risk.

Figure 3.1 ***Credit Spread Term Structure***

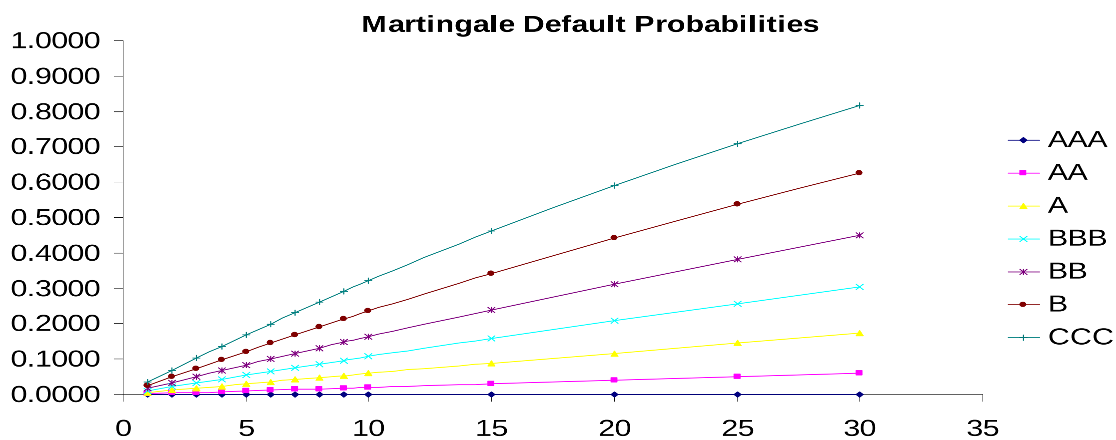


Figure 3.1 shows the default probabilities for various credit rating classes calculated in excel spreadsheet. See appendix A for spreadsheet used to generate one period transition matrix.

In the following section, Duffie and Singleton model will be used to drive a theoretical and arbitrage free credit default swap (CDS) premium. We will also implement other numerical models using binomial tree to drive non-arbitrage CDS prices.

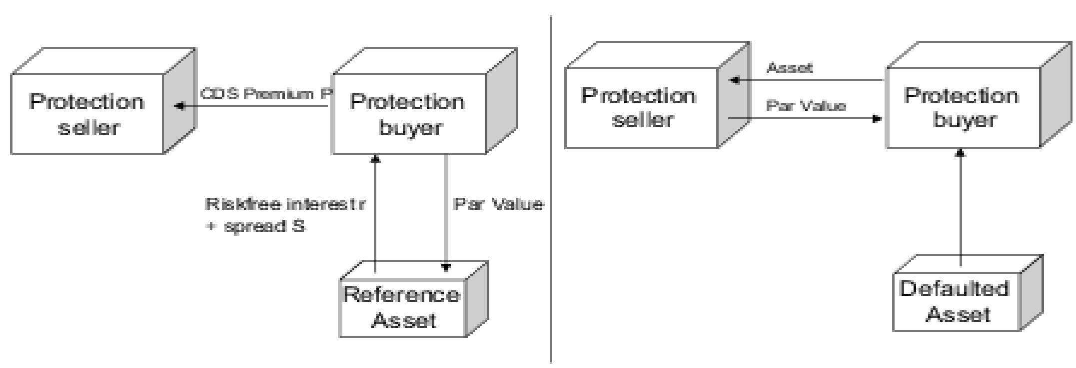
5.1 *Pricing Credit Default Swaps*

Credit default swaps (CDS) are a form of insurance against possible default of a reference issuer, or a bond issued by this issuer. The protection seller promises to compensate the protection buyer in the event of default of the reference issuer. In return, the protection buyer pays a constant periodic payment, which terminates at the earlier of the CDS maturity or a default event.

There are number of variations on the standard credit default swap. In *binary credit default swap*, the payoff in the event of default is a specific dollar amount. In a *basket credit default swap*, a group of reference entities are specified and there is a payoff when the first of these reference entities defaults. In a *contingent credit default swap*, the payoff requires both a credit event and additional trigger. The traditional trigger might be a credit event with respect to another reference entity or a specified movement in some market variable.

Several papers have addressed the theoretical pricing of credit derivatives during the last few years. Longstaff Schwartz (1995) present the pricing of credit spread options based on exogenous mean-reverting process for credit spreads. Duffie (1999) presents a simple argumentation for the replication of as well as a simple reduced form model of the instrument. In the this section, we introduce a reduced-form type pricing model developed by Hull and White (2000), where they calibrate their model based on the traded bonds of the underlying on a time series of credit default swap prices on one underlying. Like most other approaches, their model assumes that there are no counter party default risk. Default probabilities, interest rates, and recovery rates are independent. Finally, they also assume that the claim in the event of default is the face value plus accrued interest. Consider the valuation of a plain vanilla credit default swap with 1$ notional principal. Using the notations below, we proceed to show the reduced-form pricing model.

Figure 3.2 Payment structure of a CDS before and in the event of default



Notations

$P(0,T) =$ Price today of a \$1 risk-free discount bond maturing at time T
$C_R(T) =$ Par risky coupon rate for maturity T, in percent
$q(t) =$ Default probability density at time t, conditional on no prior default
$Q(t) =$ Cumulative default probability density up to time t

$$Q(T) = \int_0^T q(t)dt$$

$R =$ Recovery rate: fractional amount of bond value recovered on default.
$AI(t) =$ Accrued interest function, based on 1% per annum coupon.

$$AI(t) = \begin{cases} t - t_{i-1} & , t_{i-1} < t < t_i \\ 0 & , t = t_i \end{cases} \qquad \text{where } i \text{ is such that } t_{i-1} < t \leq t_i$$

Note: for simplicity of notations, we assume in the following equations that bond coupon payment dates and premium payment dates coincide.
A Credit Default Swap (CDS) provides protection against default of a reference issuer. The buyer of the protection pays a premium in the form of regular fixed payments S (%, annualized) for the duration of the protection period, or up to a default event. The protection seller will pay in the event of default of the reference issuer the difference between par and the post-default value of the bond.

The expected present value of the "premium" leg of the CDS is

$$S\left[\sum_{j=1}^{m}\{1 - Q(t_j)\}P(0,t_j)\Delta t_j + \int_0^T q(t)AI(t)P(0,t)dt\right] \tag{38}$$

The two terms correspond, respectively, to premium payments (made if default has not occurred) and payments of accrued premium (if default has occurred). If we assume that in the event of a future default, the recovered amount is R times par plus accrued interest, the expected present value of the "protection" leg of the CDS is

$$\int_0^T q(t)[100 - R\{100 + AI(t)C_R(T)\}]P(0,t)dt \tag{39}$$

The market value of a CDS is the difference between the two legs.

At initiation of a CDS, the premium (the CDS spread) is set to the value $S = S_{CDS}(T)$ such that the two legs of the CDS are equal, and the CDS has zero initial value. Solving for the spread:

$$S_{CDS}(T) = \frac{\int_0^T q(t)\left[100 - R\{100 + AI(t)C_R(T)\}\right]P(0,t)dt}{\sum_{j=1}^{m}\{1 - Q(t_j)\}P(0,t_j)\Delta t_j + \int_0^T q(t)AI(t)P(0,t)dt} \qquad (40)$$

This equation gives the value of a par CDS spread, with the default probability curve used as an input. Conversely, if we have a curve of par CDS spreads, we can use a bootstrap procedure to infer the default probability curve.

Note: if we do not want to include accrued interest in the default claim, we set $AI(t) \equiv 0$.

The variable $S_{CDS}(T)$ is referred to as the credit default swap spread or CDS spread. The formula at *(40)* is simple and intuitive for developing an analytical approach for pricing credit default swaps because of the assumptions used. The spread $S_{CDS}(T)$ is the payment per year, as a percent of notional principal for newly issued credit default swap contract. Table 1.5 shows the market value of $S_{CDS}(T)$ for a list of reference names. For example the quoted CDS bid/ask spread for a maturity of 5 years for France Telecommunications is: 38/44 basis points for bid and ask respectively.

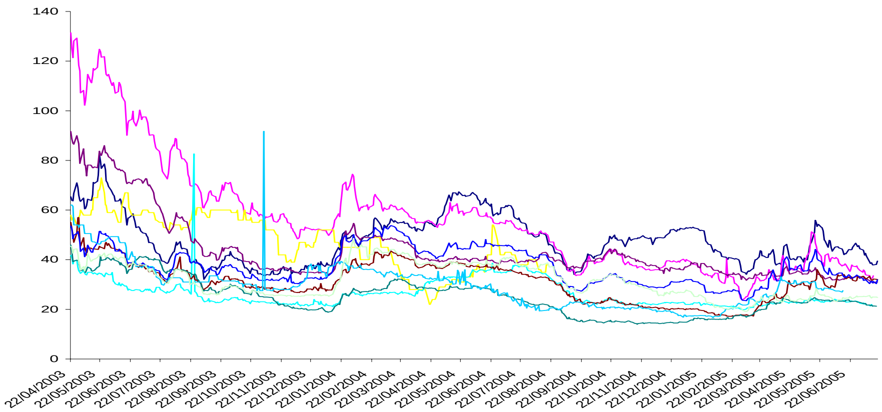
Table 1.5 CDS quotes: Telecoms and Electronics – Banco Bilbao

Telecoms & Elec	Rating		1 year		3 year		5 year		10 year	
Electronics	Moody's	S&P	bid	ask	bid	ask	bid	ask	bid	ask
BRITISH TEL	Baa1	A-	7	13	19	29	38	44	69	79
DEUTSCHE TEL	A3	A-	6	12	17	27	31	37	56	66
ELECTROLUX	Baa1	BBB+	9	15	24	34	43	53	68	78
FRANCE TELECOM	A3	A-	7	13	21	31	38	44	67	77
KPN	Baa1	A-	7	13	20	30	35	41	61	71
MMO2	Baa2	BBB	10	16	24	34	42	48	83	93
NOKIA			6	12	11	21	19	29	38	48
OTE	A3	BBB+	7	13	19	29	36	42	53	63
PHILIPS	Baa1	A-	7	13	16	26	25	35	46	56
PORTUGAL TEL	A3	A-	5	11	16	26	29	35	53	63
SIEMENS	Aa3	N.A.	4	10	4	14	18	28	35	45
STM	N.A.	BBB+	5	11	15	25	28	38	45	55
TDC A/S	Baa1	BBB+	31	37	44	54	72	78	124	134
TELECOM ITALIA	Baa2	BBB+	9	15	27	37	48	54	80	90
TELEFONICA	A3	A	6	12	15	25	31	36	54	64
TELENOR	A2	A-	5	11	14	24	25	31	45	55
TELIA SONERA	A2	A	5	11	14	24	25	31	45	55
TELSTRA	A1	A+	5	11	8	18	19	25	30	40
THOMSON	N.A.	BBB+	7	13	22	32	34	40	55	65
VODAFONE	A2	A	3	9	12	22	23	29	41	51

Source: Bloomberg LP. July 2005

To implement the above model in order to approximate the quoted market prices, one need to link the rates observed in the credit protection market and the corporate bond market via probabilities of default of the issuer. The input used to price the CDS contract should be selected from a range of market observed yield curves which should include; a curve of CDS spreads, an issuer (credit-risky) par yield curve, and default probability curve. The assumptions based on the independence of recovery rates default probabilities and interest rates may not hold completely in practice since high interest rates may cause companies to experience financial difficulties and default or administration, and as a result of this default probabilities would increase. Thus, a positive relation between interest rates and default probabilities may be associated with high discount rates for the CDS payoffs, and this would have the effect of reducing the CDS spread. Nevertheless, the Hull-White approach presents a neat and intuitive approach that allows for a closed-form pricing approach for credit default swap, calibrating market data.

Figure 4.1 ***Historical CDS for selected Telecoms reference names (2003-2005)***



Source: Bloomberg LP. July 2005
Reference names: British Tel. Deutsche Tel. France Tel. Nokia, Telefonica, Vodafone, Telenor

As an extension of the above model, Hull and white (2001) investigate the impact of counterparty default risk on the value of vanilla CDS. They find that this impact is small when the credit quality correlation between the counterparty and the reference entity is zero. It increases as the correlation increases and the creditworthiness of the counterparty declines.

5.3 Relating risky par rates & default probabilities

We assume that if a bond defaults, the amount recovered is a fraction R of the par value of the bond plus accrued interest. Recovery rates are usually reported as the ratio of the post-default value of the bond and its par value.

The equation relating the risk-free discount curve $P(0,t)$, the risky par rate $C_R(T)$, the recovery rate R, and the default probability curve $Q(t)$ is

$$100 = C_R(T)\left[\sum_{i=1}^{n}\{1-Q(t_i)\}P(0,t_i)\Delta t_i + \int_0^T q(t)AI(t)P(0,t)dt\right] + 100\{1-Q(T)\}P(0,T)$$
$$+\int_0^T q(t)R\{100 + AI(t)C_R(T)\}P(0,t)dt \qquad (41)$$

The first term on the RHS is the sum of all coupons paid assuming that default has not occurred before their payment times. The second term takes accrued interest into account: if default occurs in the middle of a coupon payment period, interest accrued on the coupon is paid. The third term is the bond face value, assuming that the issuer had not defaulted before maturity. The fourth term sums recovered values (R times par plus accrued interest) assuming default before maturity of the bond. All future payments are discounted using the risk-free discount rates. Since this is a par risky bond, the four RHS terms sum up to 100.

If the default probability curve is known, we can compute the risky par curve directly. If the risky par curve is known, we can infer the default probability curve using a bootstrap procedure.

5.4 CDS model using stochastic interest rate and intensity process

In this section, we present similar closed-form model for valuing credit default swaps within the reduced-from framework of Duffie (1998), Lando (1998), Duffie and Singletob (1998), and others. The default intensity is modeled as square-root process and explicit solution of credit default swap premia is given. Following standard notion, let r_t denote the risk-free rate and λ_t the intensity of the Poisson process governing default. Both r_t and λ_t are stochastic and are assumed to follow independent processes. In the event of default, the bondholders are assumed to recover a fraction $1-w$ of the par value of the bond. The value of risk-free zero-coupon bond $P(0,T)$ with maturity of T is given by;

$$P(0,T) = E\left[\exp\left(-\int_0^T r_t dt\right)\right] \tag{42}$$

The risk-neutral dynamics of the intensity process λ_t is given by

$$d\lambda = (\alpha - \beta\lambda)dt + \sigma\sqrt{\lambda}\, dZ \tag{43}$$

Where $\alpha,\ \beta,$ and σ are positive constants, and Z_t is a standard Brownian motion. These dynamics allow for both mean reversion and conditional heteroskedasticity in corporate spreads and guarantee that the intensity process is always nonnegative. Given these dynamics, the probability that default has not occurred by time T is given by;

$$\exp\left(-\int_0^T \lambda_t\, dt\right) \tag{44}$$

And the density function for the time until default is given by

$$\lambda_t \exp\left(-\int_0^t \lambda_s\, ds\right)dt \tag{45}$$

From Duffie (1998), Lando (1998), Duffie and Singleton (1999), the value of corporate bonds and the premium and the protection leg of credit default swap can be expressed as expectations under the risk-neutral measure. Letting c denote the coupon rate of defaultable bond, then the price of the bond which is a function of *CB(c,w,T)*[14] can be expressed as:

$$P_{0,T}^{d} = E\left[c\int_0^T \exp\left(-\int_0^t r_s + \lambda_s ds\right)dt\right] + E\left[\exp\left(-\int_0^T r_t + \lambda_t dt\right)\right]$$
$$+ E\left[(1-w)\int_0^T \lambda_t \exp\left(-\int_0^s r_s + \lambda_s ds\right)dt\right]. \qquad (46)$$

The first term in equation (46) represents the present value of the coupon portion of the bond, the second term represents the present value of the promised principal payment, and the third term is the present value of the recovery payments in the event of default.
As before, let *s* denote the premium paid by the buyer of default protection. The present value of the premium leg of a credit default swap *P(s, T)* can be expressed as

$$P(s,T) = E\left[s\int_0^T \exp\left(-\int_0^t r_s + \lambda_s ds\right)dt\right] \qquad (47)$$

And the value of the protection leg *PR(w,T)* can be expressed as

$$PR(w,T) = E\left[w\int_0^T \exp\left(-\int_0^t r_s + \lambda_s ds\right)dt\right] \qquad (48)$$

As before, we solve for *s* by setting the value of the protection and premium legs equal to each other. We then have

$$s = \frac{E\left[w\int_0^T \lambda_t \exp\left(-\int_0^t r_s + \lambda_s ds\right)dt\right]}{E\left[\int_0^T \exp\left(-\int_0^t r_s + \lambda_s ds\right)dt\right]} \qquad (48)$$

To provide some intuition about the credit default swap market, Duffie (1990) shows that premium equals the fixed spread over the risk-free rate that a corporate floating rate note would need to pay to be able to sell at par. Thus if both a firm and the treasury issued floating rate notes tied to the risk-free rate, the fixed spread between rates paid by the floating rate notes would equal the credit default premium *s*. This result is however not the case for the yield spread between corporate and treasury fixed rate bonds. Longstaff. F.A., Mithal S. and Neis E. (2003) apply the closed form solution is equation (48) and fit the model to the prices of corporate bonds. They solve for the premium implied by the model. The model implied values of the premia are then compared with the actual market credit default swap premia.

[14] This notation is also similar to previous notion for defaultable bond: $P_{t,T}^{d}$

Longstaff. F.A., Mithal S. and Neis E. (2003) used credit default swap data for 5 year contract and corresponding bond prices provided by Citigroup for 68 firms for the period of march 2001 and October 2002. They estimated the dynamics of the intensity process for each of the firm as well as all the intensity parameters in order to estimate default swap prices for these firms. Their result showed wide variation in credit default swap premia, both overtime and across firms. See table 1.6.

Table 1.6 Summary Statistics for the Differences Between Model Implied and Market Credit-Default Swap Premia. This table reports summary statistics for the differences between the premia implied by the fitted credit model and market premia for the indicated firms. Differences are expressed in basis points. Averages reported at the bottom of the table are cross-sectional averages of the indicated summary statistics taken over all firms.

Sector	Firm	Average Difference	*t*-Statistic	Min.	Max.	Serial Corr.
Financial	AON	71.5	5.2	−254.7	171.8	0.83
	Bank of America	67.3	31.4	35.2	104.0	0.91
	Bear Stearns	81.0	64.0	55.2	107.0	0.65
	Citigroup	61.4	37.6	28.9	90.2	0.83
	Countrywide Cr.	54.4	18.8	−18.9	88.8	0.57
	CIT Group	19.6	2.3	−220.3	93.1	0.41
	Capital One	31.8	2.0	−567.0	153.3	0.37
	GE Capital	40.1	15.7	−3.4	98.7	0.90
	Goldman Sachs	79.6	49.7	45.6	111.1	0.85
	Household Fin.	31.1	5.9	−93.2	89.3	0.66
	JP Morgan Chase	76.6	48.6	40.4	102.1	0.74
	MBNA	73.6	15.2	−23.3	122.4	0.67
	Lehman Brothers	73.2	45.8	41.0	103.7	0.81
	Merrill Lynch	59.1	30.6	25.5	87.9	0.87
	Morgan Stanley	75.0	58.5	45.5	97.2	0.80
	Bank One	68.6	45.4	49.6	92.6	0.82

Table 1.6[15] shows the empirical result from Longstaff. F.A., Mithal S. and Neis E. (2003) where summary statistics for the difference between the model implied and the market credit default swap premia are reported. These summary statistics include the average differences with their associated t-statistics, the minimum and maximum values of the difference, and the serial correlation of the difference. One of the most striking result from their investigation is that the average difference or pricing error is positive for all the firms (68) in the sample. Thus, the premia implied by fitting the model to the market prices of corporate bonds are all greater on average then the actual credit default swap premia observed in the market. They show that all of the average differences are highly statistically significant based on their t-statistics.

[15] We show only summary statistics for companies in the financial sector.

Figure 4.2 ***Histogram- distribution of average premium differences across firms.***

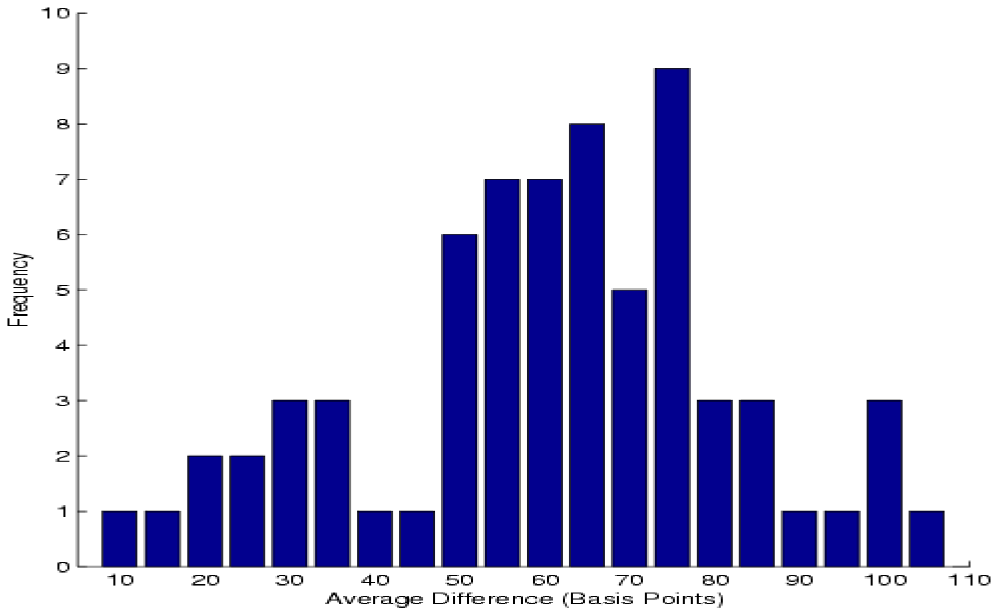


Fig. 4.2. Distribution of the average difference between the implied and market credit-default swap premia. For each firm in the sample, the average difference is calculated as the time-series average of the difference between the implied credit-default swap premium and the market credit-default swap premium. The plot shows a cross-sectional distribution of an average difference for 68 firms in the sample. The cross-sectional mean of the average difference is 60.8 basis points. The standard deviation of the average difference is 21.2 basis points.

These result strongly suggest that the cost of credit protection in the credit default swap market is significantly less than the cost implied from the corporate bond prices. Because of the cross sectional variations in the differences between implied and market premia, it is possible that other factors may be effecting the cost of protection.

In this section, we list a number of suggested factors that may contribute to the significant differences between market observed CDS prices and modeled CDS prices.

Differences in Taxation:

The differences in taxation between corporates and treasuries might explain a significant portion of the yield spread. Credit swap premium should reflect only the actual risk of default on the underlying bonds. Thus, if the spread between corporates and treasuries are partly tax related and partly default related, then this portion of the spread should not be incorporated into the credit default swap premium.

Differences in liquidity:

If corporate bonds are less liquid then treasury bonds, then corporate bond spreads could also include liquidity spread. Thus, the liquidity of corporate bonds should not affect the cost of credit protection in the CDS market. This implies that if corporate bond yields include liquidity component, then the credit default swap premia should be less then the premia implied from corporate bonds. This is consistent with the result presented in table 1.6.

Modeling error:

Another possible factor due to premia differences may be simply model error. That is, some key feature of the data is being missed by the model used to estimate the implied credit default swap premium from corporate bond prices.

Figure 4.4.1 ***Market vs implied CDS (Enron).***

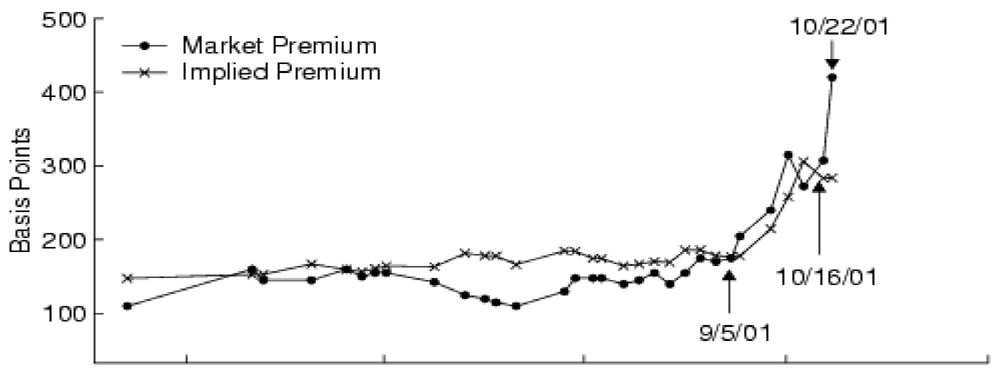


Fig. 4.4.1 Enron's market credit-default swap premium, implied credit-default swap premium, and stock price. Figure 4.4.1 shows Enron's market and implied credit-default swap premia between December 5, 2000 and October 22, 2001. The dates on the figure 4.4.1 and 4.4.2 shows chronology of some of the events leading up to Enron's bankruptcy. After rating downgrade of from credit rating class B to CC by S&P's from November 28, 2001 and November 30, 2001. Enron Filed for bankruptcy and defaulted on its debt on December 2, 2001. Near the beginning of 2001, the model and the market price were close to each other. During the middle of the year, however, the implied premium is approximately 50 basis points higher than the market premium. On average the two premia are quite close.

Figure 4.4.2 ***Historical stock prices (Enron).***

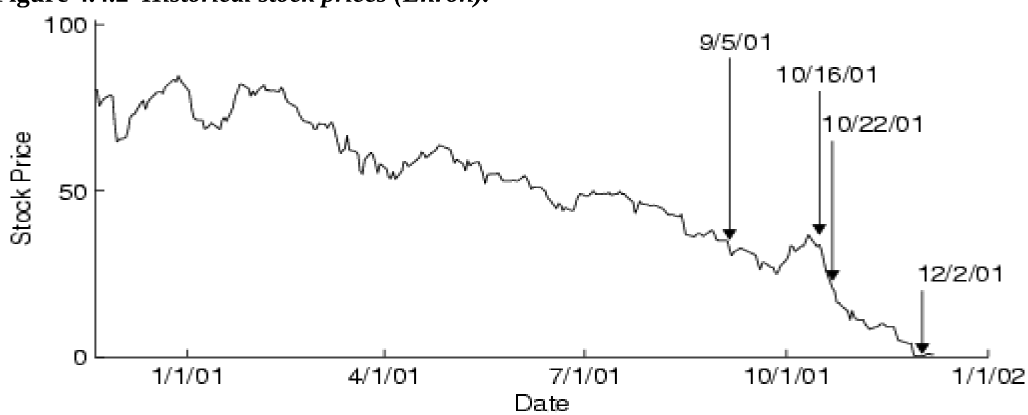
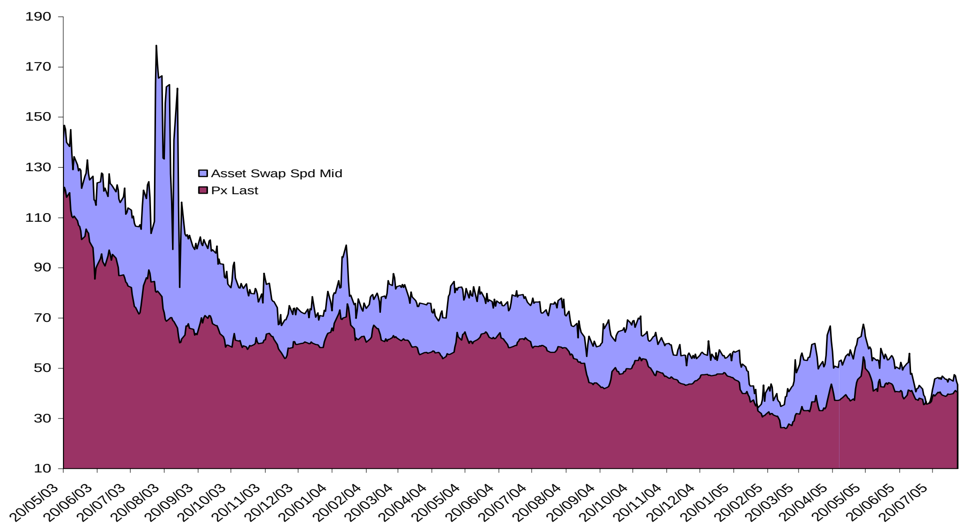


Figure 4.4.2 shows Enron's stock price between December 5, 2000 and December 7, 2001. The arrows on each plot indicate the dates of important corporate events. Enron filed for bankruptcy on December 2 2001.

Asset swap spread:

Market practitioners relate the cost of credit protection to the spread between corporate yields and swap yields. The credit default swap premium is related to the asset swap spreads, and the difference between the CDS premium and the asset swap spread is referred to as the basis[16].

Figure 4.5 ***Historical mid Asset swap spread vs CDS spread (2003-2005)***



Source: Bloomberg LP. July 2005

Using Bloomberg data for a single reference entity 5 year credit default swap premium, and a mid asset swap spread from the reference obligation of the credit default swap contract, figure 4.5 show significant difference in basis point. This result is also consistent with the result reported by Longstaff. F.A., Mithal S. and Neis E. (2003).

Longstaff. F.A., Mithal S. and Neis E. (2003) used the swap curve as a benchmark curve to determine the discount function. From their result, the use of the swap curve in estimating the discount function could not account for the large cross-sectional differences across firms. However, the average differences between implied and market premia across all firms was only 3.9 basis point according to the authors.

Default risk from counter party:

Another reason that could explain why the market observed CDS premia are lower than the implied CDS is that the firm selling credit protection might enter financial distress itself. The price of the premium from the buyers point of view should not be worth as much if there is a default correlation between the protection seller and the reference entity.

[16] for more in-depth analysis of asset swap pricing and basis arbitrage, see; Frank J. Fabozzi, Moorad Choudhry "Credit derivatives; Instruments, application and pricing."

There are several other factors that may contribute to the differences between model implied CDS premiums and market quoted CDS premiums. Factors such as the cost of shorting corporate bonds could be considered. It is however beyond the scope of this paper to examine all of these factors. It is possible that further improvements of the implemented models or alternative models in the future as a result of further research may reduce the CDS premium variations observed in most empirical work.

6.1 Kettunen, Ksendzovsky, and Meissner (KKM) model (2003)

In this section, we use an alternative model to price CDS using KKM model[17]. Kettunen, Ksendzovsky, and Meissner derive the default swap premium with a combination of two easily implementable discrete binomial trees. One tree represents the default swap premium, and the other the default swap payoff incase of default.

1. KKM model excluding counterparty default risk

Notations:

λ_t^r : risk-neutral probability of default of reference entity r during $(t, t+1)$

s_t : default swap premium to be paid at time t

N : notional

τ_t : time between 0 and time t, expressed in years

$\Delta\tau_t$: time between t and $t+1$, expressed in years

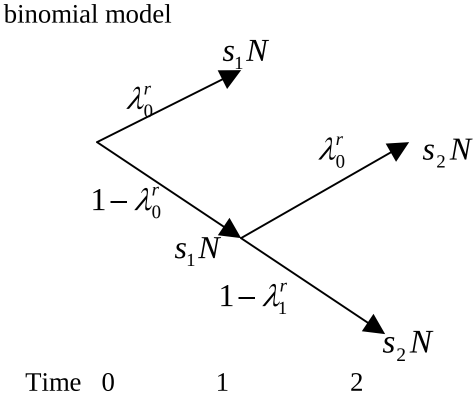
RR : recovery rate

a : accrued interest from last coupon date until the default date

r : the risk-free rate for the period $(0, t+1)$

We use a simple binomial tree where the premium is paid at the end of default period, where t represents the CDS payment dates.

Figure 6.1 Discrete time binomial model



[17] See Kettunen, J., D. Ksendzvosky, and G.Meissner, "pricing default swaps including reference asset-counterparty default correlation," Hawaii Pacific University working paper, 2003.

The present value of the default swap premium payments from figure 6.1 is given by:

$$\left[\lambda_0^r s_1 N + \left(1-\lambda_0^r\right) s_1 N\right] e^{-r_0\tau_1} + \left\{\left(1-\lambda_0^r\right)\left[\lambda_1^r s_2 N + \left(1-\lambda_1^r\right) s_2 N\right]\right\} e^{-r_0\tau_1} \qquad (49)$$

Cancelling several terms in (49) and generalizing for T period, the present value of the default swap premium is given by:

$$s_1 N e^{-r_0\tau_1} + \sum_{t=2}^{T}\left[s_1 N e^{-r_t-\tau_1}\prod_{u=0}^{t-2}(1-\lambda_u^r)\right]. \qquad (50)$$

The above tree can be extended to more default period and the pricing formula will become more complicated. For example using 4 period tree, the present value of the swap premium can be written as:

$$\begin{aligned}&\left[\lambda_0^r s_1 N\Delta\tau_0 + \left(1-\lambda_0^r\right) s_1 N\Delta\tau_0\right] e^{-r_0\tau_1}\\ &+ \left\{\left(1-\lambda_0^r\right)\left[\lambda_1^r s_1 N\Delta\tau_1 + \left(1-\lambda_1^r\right) s_1 N\Delta\tau_1\right]\right\} e^{-r_1\tau_2}\\ &+ \left\{\left(1-\lambda_0^r\right)(1-\lambda_1)\left[\lambda_2^r s_2 N\Delta\tau_2 + \left(1-\lambda_2^r\right) s_2 N\Delta\tau_2\right]\right\} e^{-r_2\tau_3}\\ &+ \left\{\left(1-\lambda_0^r\right)(1-\lambda_1)(1-\lambda_2)\left[\lambda_3^r s_2 N\Delta\tau_3 + \left(1-\lambda_3^r\right) s_2 N\Delta\tau_3\right]\right\} e^{-r_3\tau_4} \qquad (51)\end{aligned}$$

Setting the swap premium s constant in time, i.e. $s_1 = s_2 = s_3$ ...., and canceling several terms in (51), we get for time T periods:

$$s_1 N\Delta\tau_0 e^{-r_0\tau_1} + \sum_{t=2}^{T}\left[s_1 N\Delta\tau_{t-1} e^{-r_t-\tau_1}\prod_{u=0}^{t-2}(1-\lambda_u^r)\right]. \qquad (52)$$

Incorporating the payoff from the default swap seller to the protection buyer in event of default as usual, this is defined again as: $N(1-RR-RRa)$ where all the parameters are as specified above. The present value of the expected payoff is given by:

$$\begin{aligned}&\lambda_0 N(1-RR-RRa)e^{-r_0\tau_1} + (1-\lambda_0)\lambda_1 N(1-RR-RRa)e^{-r_1\tau_2}\\ &+ (1-\lambda_0)(1-\lambda_1)\lambda_2 N(1-RR-RRa)e^{-r_2\tau_3} \ldots. \qquad (53)\end{aligned}$$

Generalizing (53), we get the present value of the expected payoff:

$$\lambda_0 N(1-RR-RRa)e^{-r_0\tau_1} + \sum_{t=2}^{T}\left[N(1-RR-RRa)\lambda_{t-1}e^{-r_{t-1}\tau_1}\prod_{u=0}^{t-2}(1-\lambda_u^r)\right]. \qquad (54)$$

Combining equation (52), (53), and (54), the present value of the default swap from the buyers viewpoint is derived:

$$\lambda_0 N(1-RR-RRa)e^{-r_0\tau_1} + \sum_{t=2}^{T}\left[N(1-RR-RRa)\lambda_{t-1}e^{-r_{t-1}\tau_1}\prod_{u=0}^{t-2}(1-\lambda_u^r)\right].$$

$$-sN\Delta\tau_0 e^{-r_0\tau_1} + \sum_{t=2}^{T}\left[s_1 N\Delta\tau_{t-1}e^{-r_t-\tau_1}\prod_{u=0}^{t-2}(1-\lambda_u^r)\right]. \quad (55)$$

Setting (55) to zero and solving for *s* (credit default swap premium), we get:

$$s = \frac{\lambda_0 N(1-RR-RRa)e^{-r_0\tau_1} + \sum_{t=2}^{T}\left[N(1-RR-RRa)\lambda_{t-1}e^{-r_{t-1}\tau_1}\prod_{u=0}^{t-2}(1-\lambda_u^r)\right].}{N\Delta\tau_0 e^{-r_0\tau_1} + \sum_{t=2}^{T}\left[s_1 N\Delta\tau_{t-1}e^{-r_t-\tau_1}\prod_{u=0}^{t-2}(1-\lambda_u^r)\right].} \quad (56)$$

As before, *s* represents the fair or the mid market default swap premium implied from the above model since it gives the swap a zero value at the interception of the contract. In other words, this is neither in-the-money nor out-of-the money from both seller and buyers viewpoint.

We apply the above equation to compute the default swap premium using the following data: Given a notional value of $ 1,000,000, recovery rate of 40% and CDS contract with maturity of 1 year with annual payment. Default probability of 10% and 30 percent for period one and two. Accrued interest of 1% and 4% for both periods respectively. we plug these data into formula (56) using excel. The computed CDS premium is 22.67%. Appendix B shows a print out of the result from excel spreadsheet.

7.1 Structural versus Reduced-form models

Jarrow and Protter (2004) compared structural versus reduced form credit risk models from an information based perspective. According to the authors, difference between these two model types can be characterized in terms of the information assumed known by the modeler. Structural models assume that the modeler has the same information set as the firm's manager—complete knowledge of all the firm's assets and liabilities. In most situations, this knowledge leads to a predictable default time. In contrast, reduced form models assume that the modeler has the same information set as the market—incomplete knowledge of the firm's condition. In most cases, this imperfect knowledge leads to an inaccessible default time. Jarrow and Potter argue that the key distinction between structural and reduced form models is not whether the default time is predictable or inaccessible, but whether the information set is observed by the market or not.
If one is interested in pricing a firm's risky debt or related credit derivatives, then reduced form models are the preferred approach. There is consensus in the credit risk literature that the market does *not* observe the firm's asset value continuously in time. This implies, that the simple form of structural models illustrated above does not apply. In contrast, reduced form models have been constructed, purposefully, to be based on the information available to the market.

8.1 Conclusion

This paper introduces the existing credit risk models and their applications to price the premiums in credit default swaps (CDS) contract. Both structural and reduced-form models such as Merton's model and extensions as well as intensity based models are introduced. we examined the difference between model generated CDS prices using both Merton's model and intensity based model such as the model proposed by Duffie and Singleton. according to Longstaff. F.A., Mithal S. and Neis E. (2003) , there is a clear evidence that the implied cost of credit protection is significantly higher in the corporate bond market for all the firms they used in their sample. Possible explanations for the higher cost of credit protection implied by corporate bonds could be due to number of factors including tax issues, liquidity issues, asset pricing, the cost of shorting corporate bonds, or model error due to missing out relevant data.

Appendix A

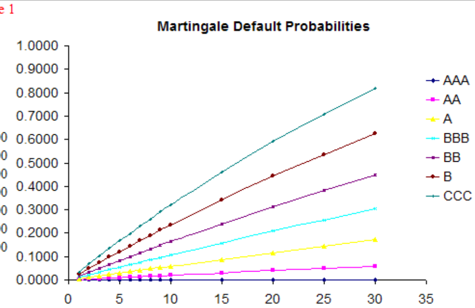
Jarrow-Lando-Turnbull 1997 Credit Risk Model

Input cells on green background. Press 'calculate' on cell c58 to derive the martingale default probabilities in figure 1

Hide Data

Average 1-year transition probabilities

	Bond Rating Class									
	AAA	AA	A	BBB	BB	B	CCC	D	NR	
AAA	0.8546	0.0945	0.0077	0.0019	0.0029	0.0100	0.0100	0.0100	0.0084	1.0000
AA	0.0084	0.8787	0.0729	0.0097	0.0028	0.0028	0.0100	0.0100	0.0047	1.0000
A	0.0009	0.0282	0.8605	0.0628	0.0098	0.0044	0.0100	0.0100	0.0134	1.0000
BBB	0.0006	0.0041	0.0620	0.7968	0.0609	0.0151	0.0017	0.0043	0.0545	1.0000
BB	0.0004	0.0020	0.0071	0.0649	0.7012	0.0942	0.0115	0.0218	0.0969	1.0000
B	0.0000	0.0017	0.0027	0.0058	0.0451	0.7196	0.0380	0.0598	0.1273	1.0000
CCC	0.0000	0.0000	0.0102	0.0102	0.0179	0.0665	0.5729	0.2046	0.1177	1.0000



Zero-coupon bond price (not used in the Martingale Default Probability Derivation)

Year	Year													
	1	2	3	4	5	6	7	8	9	10	15	20	25	30
AAA	96.9215	92.5014	87.1225	81.2513	76.2492	70.8646	65.1637	60.8654	57.0747	52.7439	49.3630	45.9904	42.6181	39.2471
AA	96.9064	92.0011	86.7370	80.9128	75.9209	70.5794	64.9239	60.6421	56.8693	52.5555	49.1870	45.8266	42.4665	39.1076
A	96.4118	91.5915	86.1866	80.2087	75.2303	69.9297	63.9945	59.7643	55.9816	51.7065	48.3897	45.0821	41.7748	38.4689
BBB	95.3560	90.3754	84.8356	78.7447	73.7927	68.6292	62.1318	58.0049	54.2150	50.0835	46.8659	43.6595	40.4533	37.2490
BB	93.2040	87.6054	81.0866	73.9117	68.9893	63.7359	57.2526	53.4365	49.8739	46.1042	43.1381	40.1842	37.2303	34.2787
B	94.8510	86.2495	75.3329	63.6358	58.5354	51.8234	47.3623	44.2297	41.4798	38.4922	36.0173	33.5518	31.0865	28.6229
CCC	92.1060	82.3102	72.5022	56.5784	49.6692	43.8038	40.6174	37.9481	35.7206	33.3455	31.2104	29.0795	26.9488	24.8187

Risk spreads s for risky bonds

		Year													
	prem de	1	2	3	4	5	6	7	8	9	10	15	20	25	30
AAA		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
AA	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010
A	0.0020	0.0030	0.0030	0.0030	0.0030	0.0030	0.0030	0.0030	0.0030	0.0030	0.0030	0.0030	0.0030	0.0030	0.0030
BBB	0.0025	0.0055	0.0055	0.0055	0.0055	0.0055	0.0055	0.0055	0.0055	0.0055	0.0055	0.0055	0.0055	0.0055	0.0055
BB	0.0030	0.0085	0.0085	0.0085	0.0085	0.0085	0.0085	0.0085	0.0085	0.0085	0.0085	0.0085	0.0085	0.0085	0.0085
B	0.0040	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125	0.0125
CCC	0.0050	0.0175	0.0175	0.0175	0.0175	0.0175	0.0175	0.0175	0.0175	0.0175	0.0175	0.0175	0.0175	0.0175	0.0175
	yearly premium delta (%		**0.0000**												

The result on the spreads are generated using VBA code which we omit from this text due to the length of the code.

Appendix B

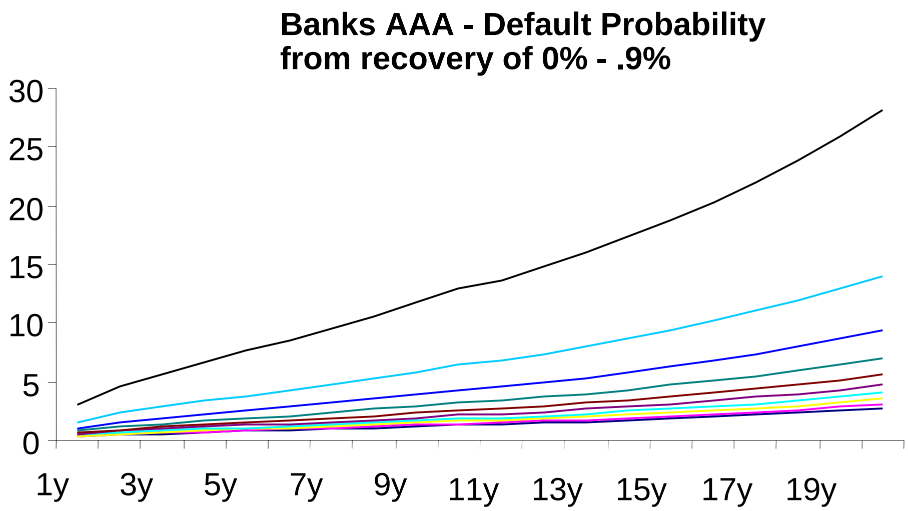
Equation 56

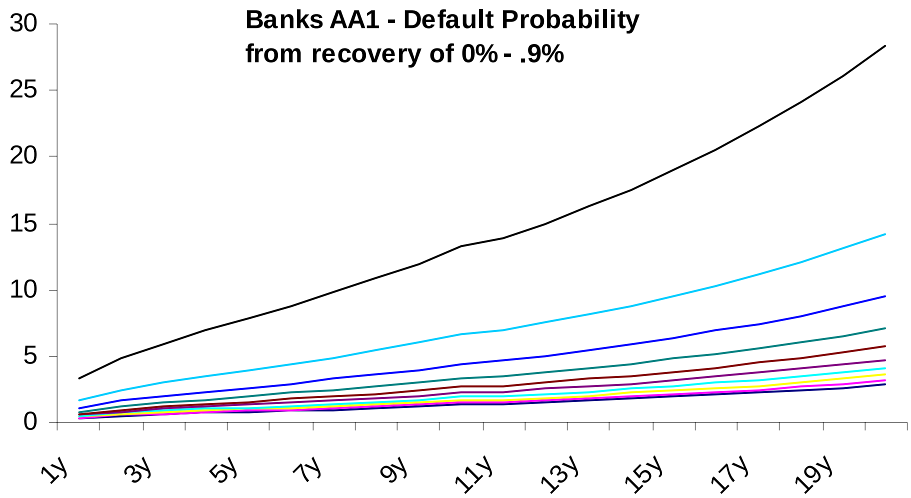
Input:

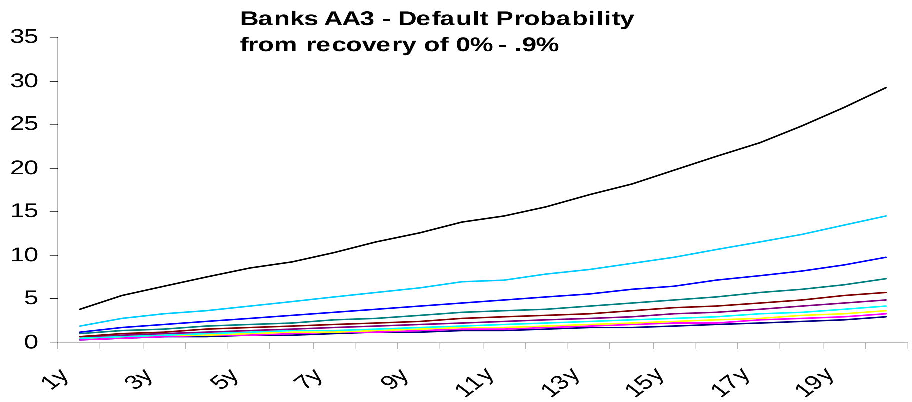
				time 1	time 2				
time 1=	0.5	year	Numerator	58,128	148,497		206,626	s=	22.67%
time 2=	1	year	Denominator	487,655	423,794		911,449		
N=	1000000								
RR=	40.00%		s=	11.92%	35.04%	46.96%			
Lambda(t0)	10.00%					0.2348			
Lambda(t1)	30.00%								
r0	5.00%						0.145238		
r1	6.00%								
a0-0.5	1.00%								
a1	4.00%								
delta thau0.5	0.50								

$$s = \frac{\lambda_0^r N(1 - RR_r - RR_r a)\, e^{-r_0\tau_1} + \sum_{t=2}^{T} [N(1 - RR_r - RR_r a)_t\, \lambda_{t-1}^r e^{-r_{t-1}\tau_t} \prod_{u=0}^{t-2} (1 - \lambda_u^r)]}{N\, \Delta\tau_0\, e^{-r_0\tau_1} + \sum_{t=2}^{T} [N\, \Delta\tau_{t-1}\, e^{-r_{t-1}\tau_t} \prod_{u=0}^{t-2} (1 - \lambda_u^r)]}$$

Appendix C[18]







[18] Source: http://www.freecreditderivatives.com

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