Introducing formal analysis of the Term Structure of Interest Rates

Learning Curve
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Our previous Learning Curve showed how one could extract spot interest rates form the prices and yields of coupon bonds, and how one could use these rates to imply rates for future time periods, so-called forward interest rates. In this article we introduce the terminology used to describe these rates that one will encounter in formal analysis.

Under the following conditions:

- frictionless trading conditions;
- competitive economy;
- discrete time economy;

with discrete trading dates of $\{0,1,2,.....,\tau\}$, we assume we have a set of zero-coupon bonds with maturities $\{0,1,2,.....,\tau\}$. The price of a zero-coupon bond at time t with a nominal value of £1 on maturity at time T (such that $T \geq t$) is denoted with the term $P(t, T)$. The bonds are considered risk-free.

The price of a bond at time t of a bond of maturity T is given by

$$P(t,T) = \frac{1}{[y(t,T)]^{(T-t)}}$$

where $y(t, T)$ is the yield of a T-maturity bond at time t. Re-arranging the above expression, the yield at time t of a bond of maturity T is given by

$$y(t,T) = \left[\frac{1}{P(t,T)}\right]^{1/(T-t)}.$$

The time t forward rate that applies to the period $[T, T+1]$ is denoted with $f(t, T)$ and is given in terms of the bond prices by

$$f(t,T) = \frac{P(t,T)}{P(t,T+1)}.$$

This forward rate is the rate that would be charged at time t for a loan that ran over the period [T, T+1].

From the above expression we can derive an expression for the price of a bond in terms of the forward rates that run from t to T-1, which is

$$P(t,T) = \frac{1}{\prod_{j=t}^{T-1} f(t,j)}.$$

This expression means:

$\prod_{j=t}^{T-1} f(t,j) = f(t,t)\cdot f(t,t+1)\ldots f(t,T-1)$, that is, the result of multiplying the rates that apply to the interest periods in index j that run from t to T-1. It means that the price of a bond is equal to £1 received at time T, that has been discounted by the forward rates that apply to the maturity periods up to time T-1.

We show how this expression is derived below:

Consider the following expression for the forward rate applicable to the period (t, t),

$$f(t,t) = \frac{P(t,t)}{P(t,t+1)}$$

but of course $P(t, t)$ is equal to 1, so therefore

$$f(t,t) = \frac{1}{P(t,t+1)}$$

which can be re-arranged to give

$$P(t,t+1) = \frac{1}{f(t,t)}.$$

For the next interest period we can set

$$f(t,t+1) = \frac{P(t,t+1)}{P(t,t+2)}$$

which can be re-arranged to give

$$P(t,t+2) = \frac{P(t,t+1)}{f(t,t+1)}.$$

We can substitute the expression for $f(t, t+1)$ into the above and simplify to give us

$$P(t, t+2) = \frac{1}{f(t,t)f(t,t+1)}.$$

If we then continue for subsequent interest periods (t, t+3) onwards, we obtain

$$P(t, t+j) = \frac{1}{f(t,t)f(t,t+1)f(t,t+2)....f(t,t+j-1)}$$

which is simplified into our result above.

Given a set of risk-free zero-coupon bond prices, we can calculate the forward rate applicable to a specified period of time that matures up to the point T-1. Alternatively, given the set of forward rates we are able to calculate bond prices.

The zero-coupon or spot rate is defined as the rate applicable at time t on a one-period risk-free loan (such as a one-period zero-coupon bond priced at time t). If the spot rate is defined by $r(t)$ we can state that

$$r(t) = f(t,t).$$

This spot rate is in fact the return generated by the shortest-maturity bond, shown by

$$r(t) = y(t,t+1).$$

We can define forward rates in terms of bond prices, spot rates and spot rate discount factors.

Example
Zero-coupon bond prices, spot rates and forward rates

Period	**Bond price [*P(0, T)*]**	**Spot rates**	**Forward rates**
0	1		
1	0.984225	1.016027	1.016027
2	0.967831	1.016483	1.016939
3	0.951187	1.016821	1.017498
4	0.934518	1.017075	1.017836
5	0.917901	1.017280	1.018102
6	0.901395	1.017452	1.018312
7	0.885052	1.017597	1.018465
8	0.868939	1.017715	1.018542
9	0.852514	1.017887	1.019267